CHAPTER 3

Rigidity of

Skeletal Structures

3.1 INTRODUCTION

The rigidity of structures has been studied by pioneering structural engineers such as Henneberg [79] and Müller-Breslau [176]. The methods they developed for examining the rigidity of skeletal structures are useful for the study of structures either with a small number of joints and members, or possessing special connectivity properties. Rigid-jointed structures (frames), when supported in an appropriate form and containing no releases, are always rigid. Therefore only truss structures require to be studied for rigidity.

Various types of methods have been employed for the study of rigidity; however, the main approaches are either algebraic or combinatoric. A comprehensive discussion of algebraic methods may be found in the work of Pellegrino and Calladine [190]. The first combinatorial approach to the study of rigidity is due to Laman [150] who found the necessary and sufficient conditions for a graph to be rigid, when its members and nodes correspond to rigid rods (bars) and rotatable pin-joints of a planar truss. Certain types of planar truss have been studied for rigidity by Bolker and Crapo [15], Roth and Whiteley [211] and Crapo [33].

Although Laman theoretically solved the problem of rigidity for planar trusses, no algorithm was given to check whether a given graph was rigid. Two combinatorial algorithms are developed by Lovász and Yemini [165] and Sugihara [227], the inter-relation of which has been shown by Tay [231]. Some studies have recently been made in the direction of extending the developed concepts for planar trusses to those of space trusses. However, the results obtained are incomplete and applicable only to special classes of space truss (see Ref. [247] as an example). Therefore, in this chapter, only planar trusses are studied.

3.2 DEFINITIONS

Rigidity of trusses can be studied at different levels. The first level is combinatorial - is the graph of joints and members (bars) correct? The second level is geometrical - is the placement of joints appropriate? The third level is mechanical - are the selected materials and methods of construction suitable? This chapter is devoted to the first level rigidity analysis of planar trusses. For this purpose, simplifying assumptions and definitions are made as follows:

Consider a planar truss composed of rigid members and pinned joints. Each joint connects the end nodes of two or more members in such a way that the mutual angles of the members can change freely if the other ends are not constrained. Such an assumption is adequate for the first level analysis of the rigidity. Let M(S) and N(S) denote the set of members and nodes of the graph model S of a truss. Denote the Cartesian coordinates of a node $n_i \in N(S)$ by (x_i,y_i). The number of members and nodes of S, as before, are also denoted by M(S) and N(S), respectively.

A member connecting n_i to n_j constrains the movement of S in such a way that the distance between these two nodes remains constant, i.e.

$$(x_i - x_j)^2 + (y_i - y_j)^2 = \text{const.} \tag{3-1}$$

Differentiating this equation with respect to time t, we get,

$$(x_i - x_j)(\dot{x}_i - \dot{x}_j) + (y_i - y_j)(\dot{y}_i - \dot{y}_j) = 0, \tag{3-2}$$

where the dot denotes the differentiation with respect to t. Equation (3-2) implies that the relative velocity should be perpendicular to the member, that is, no member is stretched or compressed. Writing all such equations for the members of S, the following system of linear equations is obtained,

$$\mathbf{Hw} = \mathbf{0}, \tag{3-3}$$

where **H** is a M(S)×2N(S) constant matrix and **w** is a column vector of unknown variables $\mathbf{w} = \{\dot{x}_1\ \dot{x}_2 \ldots \dot{x}_{N(S)} \dot{y}_1\ \dot{y}_2 \ldots \dot{y}_{N(S)}\}^t$, t denoting the transpose. A vector **w** which satisfies Eq. (3-3), is called an *infinitesimal displacement* of S. The infinitesimal displacements of S with respect to point-wise addition and multiplication by scalars, form a linear vector space $R^{2N(S)}$. The rigid body motion in a plane is a three-dimensional subspace of this linear space. The co-dimension of this subspace of rigid motions in the space of all infinitesimal motions is called the *degree of freedom* of S, denoted by f(S). The structure S is *rigid* if f(S) = 0.

As an example, consider a truss as shown in Figure 3.1. For this truss, matrix **H** and vector **w** can be written as:

$$\mathbf{H} = \begin{bmatrix} x_i - x_j & x_j - x_i & 0 & y_i - y_j & y_j - y_i & 0 \\ x_i - x_k & 0 & x_k - x_i & y_i - y_k & 0 & y_k - y_i \\ 0 & x_j - x_k & x_k - x_j & 0 & y_j - y_k & y_k - y_j \end{bmatrix},$$

and

$$\mathbf{w} = \{\dot{x}_i \quad \dot{x}_j \quad \dot{x}_k \quad \dot{y}_i \quad \dot{y}_j \quad \dot{y}_k\}^t.$$

The entries of **H** are real and linear functions of the nodal coordinates of the corresponding graph.

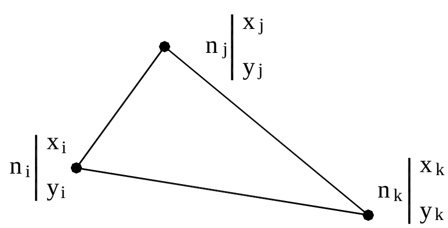
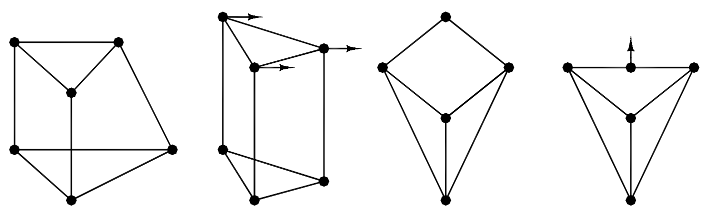


Fig. 3.1 A triangular planar truss.

It should not be thought that the rigidity of S requires Eq. (3-3) to have only the trivial solution **w** = **0**. Any rigid body motion with non-trivial **w** will also satisfy this equation. As an example, consider a translation of the entire S specified by a vector $\{a,b\}^t$, i.e. $\dot{x}_i = \dot{x}_j = \dot{x}_k = a$ and $\dot{y}_i = \dot{y}_j = \dot{y}_k = b$. Obviously **Hw** = **0** still holds, since the sum of the first (or second) three columns of **H** is zero. Therefore rank (**H**) < 2N(S). The rigid body motion subspace in plane has dimension 3, and for any truss we have rank (**H**) ≤ 2N(S) − 3. However, if rank (**H**) = 2N(S) − 3, then S is called *rigid* and for rank (**H**) < 2N(S) − 3, it is *non-rigid*. In the above example rank (**H**) = 2×3 − 3 = 3 holds, and therefore a triangular planar truss is rigid.

Now consider other examples as shown in Figure 3.2. The truss shown in Figure 3.2(a) is rigid, while the one in Figure 3.2(b) is not rigid, although their underlying graphs are the same. The assignment of velocities, indicated by arrows, forms an infinitesimal displacement because it does not violate Eq. (3-3). The nodes without arrows are assumed to have zero velocities. Similarly, though Figure 3.2(c) and Figure 3.2(d) have the same graph models, (c) is rigid but (d) is not rigid. It should be noted that an infinitesimal displacement does not always correspond to an actual movement of a mechanism. The truss (b) deforms mechanically, while truss (d) violates only Eq. (3-3).



(a) (b) (c) (d)

Fig. 3.2 Rigid and non-rigid planar trusses.

The nodes of a structure S are in *general position* if x_1, y_1, x_2, y_2, ...,$x_{N(S)}$, $y_{N(S)}$ are algebraically independent over the rational field. When the nodes are in general position, the definition of algebraic dependence shows that a subdeterminant of matrix **H** is 0, if and only if it is identically **0,** when x_1, y_1, ... , $x_{N(S)}$, $y_{N(S)}$ are considered as variables. Therefore if the nodes of S are in general position, the linear independence of the Eq. (3-3) depends only on the underlying graph, and consequently the rigidity also depends only on the graph model of the structure. From now on it is assumed that the nodes of S are in general position.

For a ball-jointed space truss, Eq. (3-3) can be written in a general form to include a third dimension z. For such a case, a rigid body motion in space is a six dimensional subspace of $R^{3N(S)}$. Therefore a space truss will be rigid if rank (**H**) = 3N(S) – 6 and non-rigid if rank (**H**) < 3N(S) – 6.

Suppose S is the graph model of a planar truss whose joints are in general position. A graph S is called *stiff* if the corresponding truss is rigid. For any X $\subseteq$ M, let $\rho_S(X)$ be the rank of submatrix of **H** consisting of the rows associated with the members of X. X is called *generically independent* if $\rho_S(X) = |X|$, and generically dependent if $\rho_S(X) < |X|$. |X| denotes the cardinality of X.

For any subset X of M(S), define,

$$\mu_S(X) = -M(X) + 2N(X) - 3, \tag{3-4}$$

where $|M(X)| = |X|$. Then the following basic theorem on rigidity can be stated:

Theorem 1 (Laman [150]): The graph S is generically independent if and only if $\mu_S(X) \geq 0$ for any non-empty subset X of M(S).

Corollary 1: S is stiff if and only if there exists $M' \subseteq M(S)$ such that $|M'| = 2N(S) - 3$ and $\mu_S(X) \geq 0$ for every non-empty subset X of M'.

Corollary 2: S is stiff and generically independent, if and only if

(a) $\mu_S(M) = 0$ and

(b) $\mu_S(X) \geq 0$ for every non-empty subset X of M(S).

Using $\gamma(S) = M(S) - 2N(S) + 3 = -\mu_S(S)$, Theorem 1 can be restated as follows:

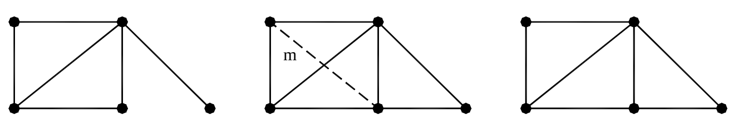
The graph S is generically independent if and only if $\gamma(S_i) \leq 0$ for every subgraph S_i of S, Figure 3.3(a). The graph S is stiff if and only if there is a covering subgraph $\overline{S}$ of S such that $\gamma(\overline{S}) = 0$ and $\gamma(S_i) \leq 0$ for every non-empty subgraph S_i of $\overline{S}$, Figure 3.3(b), which is a statically indeterminate structure.

Finally, the graph S is stiff and generically independent, if and only if

(a) $\gamma(S) = 0$ and

(b) $\gamma(S_i) \leq 0$ for every subgraph S_i of S, Figure 3.3(c), which is a statically determinate truss.

Unfortunately the application of Theorem 1 requires $2^{M(S)}$ steps to determine whether a graph is generically independent. In the next sections, two methods are described for an efficient recognition of generic independence, which were developed by Sugihara, and Lovasz and Yemini.



(a) S (b) $S = \overline{S} \cup m$ (c)

Fig. 3.3 Generically independent, stiff, generically independent and stiff graphs.

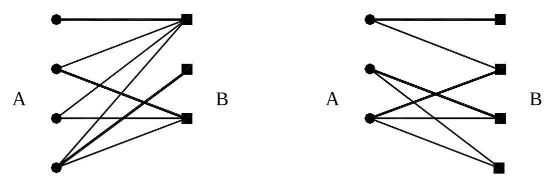
3.3 MATCHING FOR THE RECOGNITION

OF GENERIC INDEPENDENCE

Planar trusses are frequently used in structural engineering and therefore a method is presented in this section for checking the rigidity of these structures, which is suitable for both determinate and indeterminate trusses. The algorithm is a polynomial bounded one and uses complete matching of a specially constructed bipartite graph for the recognition of generic independence.

Definitions: Let B(S) = (A,E,B) be a bipartite graph with node sets A, B and member set E. A subset E' of E is called a *complete matching* with respect to A if the end nodes of members in E' are distinct and if every node in A is an end node of one member in E'.

For $X \subseteq A$, let $\Gamma(X)$ be the set of all those nodes in B that are connected to nodes in X by some members of E. As has been described in Chapter 1, a graph has complete matching, if and only if $|X| \leq |\Gamma(X)|$ for every $X \subseteq A$. Examples of matching and complete matching are depicted in Figures 3.4(a) and (b), respectively. Using these definitions, the method is described as follows:



(a) A matching. (b) A complete matching.

Fig. 3.4 Examples of matching.

Construct a bipartite graph of S. For this purpose let S be a graph with N(S) nodes and M(S) members. The corresponding node set and member set are shown with the same symbols. For each node n_i of S let p_i and q_i be two distinct symbols. Then let B(S) = (A*,E*,B*) be the bipartite graph whose node sets A* and B* and member set E* are defined as:

$$A^* = M(S),$$

$$B^* = \{p_1,q_1,\ p_2,q_2,\ \ldots,p_{N(S)},\ q_{N(S)}\ \}$$

$$E^* = \{(m,p_i),(m,q_i),(m,p_j),(m,q_j) \mid m = \{n_i,n_j\} \in M(S)\}.$$

This bipartite graph is now augmented as follows:

Let t_1, t_2 and t_3 be three distinct symbols. Then for any $1 \le i < j \le N(S)$, let $B_{ij}(S) = (A^*, E_{ij}, B^*)$ be the new bipartite graph constructed from B(S) by the addition of three nodes and three members in the following manner:

$$A_c^* = A^* \cup \{t_1,t_2,t_3\},$$

$$E_{ij} = E^* \cup \{(t_1,p_i),(t_2,q_i),(t_3,p_j)\}.$$

For any $Z \subseteq A_c^*$, denote by $\Gamma_{ij}(Z)$ the set of nodes of B^* which are connected to elements of Z by members in E_{ij}. For any $X \subseteq A_c^*$, note that $2N(X) = |\Gamma_{ij}(X)|$. Then the following theorem can be proven.

Theorem 2 (Sugihara [227]): The graph model S, is generically independent, if and only if for any i and j $(1 \le i < j \le N(S))$, $B_{ij}(S) = (A_c^*, E_{ij}, B^*)$ has a complete matching with respect to A_c^*.

Proof: Suppose S is generically independent. Let $X \subseteq M(S)$ and $Y \subseteq \{t_1,t_2,t_3\}$), and consider $Z = X \cup Y$ to be any subset of A_c^*. If $X = \emptyset$, then $\Gamma_{ij}(Z) = |\Gamma_{ij}(Y)| = |Y| = |Z|$. If $X \neq \emptyset$, then:

$$\Gamma_{ij}(Z)| \ge 2|N(X)| \ge |X| + 3 \ge |Z|, \qquad (3\text{-}5)$$

where the first inequality follows from the definition of $B_{ij}(S)$ and the second from Laman´s theorem. Then in every case, $|\Gamma_{ij}(Z)| \ge |Z|$ and hence $B_{ij}(S)$ has a complete matching with respect to A_c^*.

Now suppose for $1 \le i < j \le N(S)$, $B_{ij}(S)$ has a complete matching with respect to A_c^*. In this case $|Z| \le |\Gamma_{ij}(Z)|$ is satisfied for $Z \subseteq A_c^*$. Let X be any non-empty subset of M(S). Then N(X) contains at least two nodes, like n_k and n_l $(1 \le k \le N(S))$. Because $B_{kl}(S)$ has a complete matching, we have:

$$|X \cup \{t_1,t_2,t_3\}| \le |\Gamma_{kl}(X \cup \{t_1,t_2,t_3\})|. \qquad (3\text{-}6)$$

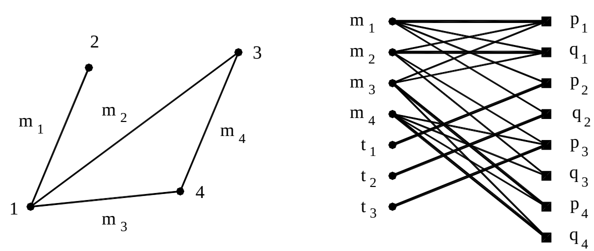
Since $\Gamma_{kl}(X \cup \{t_1,t_2,t_3\}) = \Gamma_{kl}(X)$, it follows that:

$$2|N(X)| = |\Gamma_{kl}(X)| \ge |X \cup \{t_1,t_2,t_3\}| = |X| + 3. \qquad (3\text{-}7)$$

By Theorem 1 it is concluded that S is generically independent.

A complete matching of $B_{ij}(S) = (A_c^*, E_{ij}, B^*)$ can be found by Hopcroft and Karp´s algorithm [80]. The number of $B_{ij}(S)$s is proportional to $M(S) \times M(S)$.

As an example, consider the graph S as shown in Figure 3.5(a). The bipartite graph of S is depicted in Figure 3.5(b) and a typical complete matching $B_{23}(S)$ is illustrated in bold lines. The examination of all $B_{ij}(S)$ for $1 \leq i < j \leq 4$ shows that complete matching exists and S is a generically independent graph.



(a) S. (b) $B_{23}(S)$.

Fig. 3.5 A complete matching $B_{23}(S)$ of a graph S.

The above method is quite general and it is applicable to statically determinate and indeterminate structures. Another approach for the recognition of determinate trusses is due to Lovász and Yemini [165], which is described in the following section.

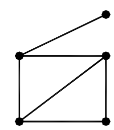
3.4 A DECOMPOSITION APPROACH FOR THE RECOGNITION OF GENERIC INDEPENDENCE

A graph S is generically independent if doubling any member of S results in a new graph, which is the union of two spanning forests. A spanning forest is the union of k trees containing all the nodes of S. This is the result of a special case of the following theorem:

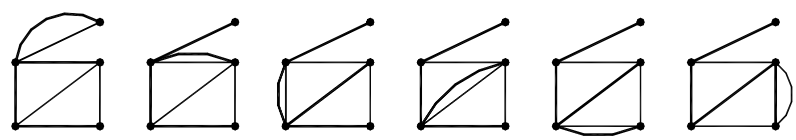
Theorem 3 (Nash Williams [179]): A graph S has a k member-disjoint spanning forest, if and only if $M(S_i) \leq k[N(S_i) - 1]$ for every partition of N(S)**.**

Consider k = 2, then $M(S_i) \leq 2N(S_i) - 2$ for every $S_i \subseteq S$. If a member is added to S_i without increasing its nodes $S_i' = S_i \cup m$, then $M(S_i') - 2N(S_i') + 3 \leq 0$, i.e. $\mu(S_i') \geq 0$ for every $S_i \subseteq S$. This verifies the above method for checking the generic independence of S.

As an example, the above method is applied to check the graph shown in Figure 3.6(a) for generic independence. It can be seen that doubling any member of S leads to a graph which can be decomposed into two forests. The members for one of these forests, which have become spanning trees, are shown in bold lines in Figure 3.6(b).



(a) A graph S.



(b) Decomposition of $S \cup m_i$.

Fig. 3.6 The generic independence check of S.

The above two seemingly different methods are mathematically inter-related. A proof of this fact can be found in Ref. [231].

There is an algorithm for controlling whether a given graph is the union of two member-disjoint forests, which is a polynomial algorithm, Clausen and Hansen [28]. For definition of a polynomial algorithm, consider a problem of size n, where n in a graph-theoretical problem can be taken as the number of members or number of nodes of a graph. An algorithm is of order n^k, denoted by $O(n^k)$ for some constant k, if its worst-case running time (or the number of operations) is equal to Cn^k, where C is a fixed constant (changes from one algorithm to another). All such algorithms are known as *polynomial algorithms*, which are considered to be efficient (tractable).

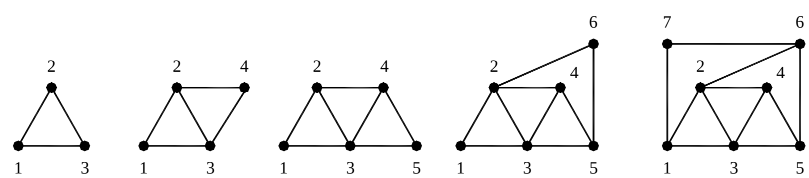
There are problems for which it is proven that no polynomial algorithms can be designed. These problems are called *NP-hard*, considered as inefficient (intractable) algorithms. The third important class of problems consists of those for which no polynomial algorithms have yet been discovered, but it is not proven that no such algorithm exists for any of them. These problems are known as *NP-complete* problems, and have the property that if a polynomial algorithm can be found for any of them, then all the others will have polynomial algorithms. There are many NP-complete problems and it could be astounding if polynomial algorithms could be found for all of them.

3.5 RIGIDITY OF PLANAR TRUSSES: SPECIAL METHODS

3.5.1 SIMPLE TRUSSES

A special type of statically determinate trusses are known as simple trusses, defined earlier in Section 2.3.2. This type of truss is reconsidered from the point of view of rigidity. A simple truss can be constructed as follows:

Start with a rigid unit such as ground or a triangle, and in each step of expansion add a star of degree 2 connected to the previously expanded part at its two free nodes. If the three nodes of the added star are not collinear, then the constructed truss is rigid and statically determinate. As an example, a simple planar truss is formed in 5 steps as depicted in Figure 3.7.



S_1 $\quad S^2 = S_1 \cup S_2$ $\quad S^3 = S^2 \cup S_3$ $\quad S^4 = S^3 \cup S_4$ $\quad S^5 = S^4 \cup S_5 = S$

Fig. 3.7 An expansion process for the formation of a simple truss.

A constructive proof can be obtained by considering S, joint by joint, in the reverse order of the expansion; i.e. 7, 6, 5 and 4. At each step two independent equilibrium equations with two unknowns, having unique bar forces, can be obtained, which provides the determinacy and rigidity of the whole structure. Mathematically, using graph theory, the following proof is obtained:

For the subgraphs used in the process of expansion we have:

$$\gamma(S_1) = 0 \,, \quad \gamma(S_i) = -1 \quad \text{for} \quad i=2,...,5,$$

and
$$\bar{\gamma}\,(A_i) = 0 - 2\times 2 + 3 = -1 \quad \text{for} \quad i=2,...,5.$$

Therefore, using the intersection theorem (Eq. (2-9)), we have:

$$\gamma(S) = 0 + 4(-1) - 4(-1) = 0.$$

However, one should show that S is also rigid. For this purpose the following theorem can be employed:

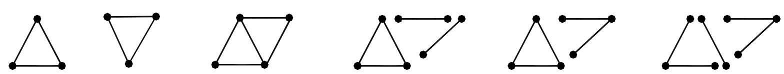
Theorem 4 (Lovász and Yemini [165]): A graph G is stiff, if and only if

$$\sum_{i=1}^{k}[2N(G_i)-3] \geq 2N(G)-3\,, \tag{3-8}$$

holds for every set of subgraphs G_i (i=1,...,k) such that $G_1 \cup G_2 \cup ... \cup G_k = G$.

It is easy to note that it would be sufficient to consider $\{G_1, G_2, ... , G_k\}$ which consist of member-disjoint spanning subgraphs. Obviously when a graph is stiff, the corresponding truss will be rigid.

Example: Consider S_1 and S_2 as shown in Figures 3.8(a-b), the union of which is S^2, Figure 3.8(c). Various decompositions of S^2 are shown in Figures 3.8(d-f). In order not to mix these subgraphs with the original ones, G_i is employed for denoting the decomposed subgraphs.



(a) S_1 (b) S_2 (c) S^2 (d) G_1=S_1,G_2,G_3 (e) G_1=S_1, G_2=S_2 (f) G_1,G_2,G_3

Fig. 3.8 Rigidity control of a simple truss.

For (d), Eq. (3-8) becomes:

$(3) + (1) + (1) \geq 2\times4 - 3$ which is true.

For (e) we have, $(3) + (3) \geq 5$ true,

and for (f), $(3) + (1) + (1) + (1) \geq 5$ which is also true.

For any other decomposition, the inequality (3-8) holds, and therefore S^2 is rigid. Similar arguments can be made for S^3, S^4 and S^5 = S of Figure 3.8 verifying the rigidity of a simple truss.

3.5.2 TRUSSES IN THE FORM OF 2-TREES

These trusses are a special case of simple trusses in that a star of degree 2 is joined to the end nodes of an expanded member. In Figure 3.7, up to S^3, this restriction is taken into account, i.e. S^3 is a 2-tree, however, S^4 and S^5 are not. Therefore, a 2-

tree is always rigid and statically determinate. As an example, a 2-tree with its interchange graph is shown in Figure 3.9. An interchange graph is a dual graph for which the node corresponding to the unbounded region and its incident members are deleted. Obviously for a 2-tree such a graph is a tree.

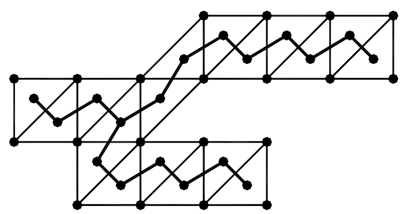


Fig. 3.9 A 2-tree and its interchange graph.

3.5.3 A γ-TREE AND ITS RIGIDITY

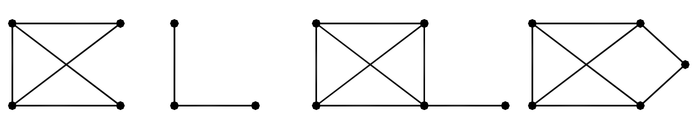
A natural extension of the idea of expansion is the use of subgraphs other than stars of degree 2. Let us consider each S_i as a generically independent subgraph of S. In the force method of structural analysis, a maximal generically independent and rigid subgraph T of S is needed, which will be described in more detail in Chapter 6. Therefore additional restrictions are considered:

$$\gamma(S_1) = 0$$

$$\gamma(S_i) = \bar{\gamma}(A_i) \quad \text{for} \quad i=2,...,q. \tag{3-9}$$

These conditions ensure $\gamma(T) = 0$; however, a γ-tree which is also rigid, is required. Thus at each step of expansion, a rigidity control is required. There are different approaches for such a control. As an example, Theorem 4 may be employed for this purpose, or a matching algorithm can be used.

Let us illustrate the process of employing Theorem 4 by a simple example. Take S_1 as a rigid graph as shown in Figure 3.10(a), and S_2 as a generically independent subgraph as Figure 3.10(b).



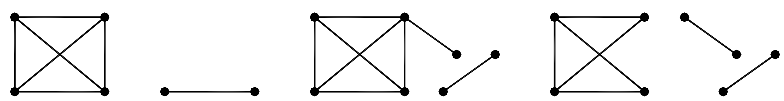
(a) S_1 (b) S_2 (c) $S_1 \cup S_2$ (d) $S_1 \cup S_2$

Fig. 3.10 Two different expansion processes.

S_2 is added to S_1 in two different manners as illustrated in Figures 3.10(c and d). Using the intersection theorem for both cases we have,

$$\gamma(S_1 \cup S_2) = 0 + (-1) - (-1) = 0,$$

while (d) is rigid, (c) is not rigid. In order to show this property, (c) and (d) are decomposed in Figure 3.11.



(a) Decomposition of (c) (b) Decomposition of (d) (c) Another decomposition of (d).

Fig. 3.11 Rigidity control by Theorem 4.

Now for decomposition of Figure 3.11(a) we have,

$$(2\times4 - 3) + (2\times2 - 3) \geq 2\times5 - 3,$$

which is false and therefore (c) is not rigid. For (d) we have:

$$(2\times5 - 3) + (2\times2 - 1) \geq 2\times5 - 3,$$

which is true. Any other decomposition leads to a true inequality and hence (d) is rigid. For example, for the decomposition of Figure 3.11(c) one obtains,

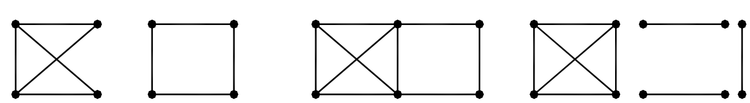
$$(2\times4 - 3) + (2\times2 - 3) + (2\times2 - 3) \leq (2\times5 - 3),$$

which is also true.

Example: Consider S_1 and S_2 as shown in Figures 3.12(a) and 3.12(b), respectively. For the decompositions shown in (d) we have,

$$(2\times4 - 3) + (2\times2 - 3) + (2\times2 - 3) + (2\times2 - 3) \geq (2\times6 - 3),$$

which is false, and the subgraph $S_1 \cup S_2$ is not rigid, though $\gamma(S_1 \cup S_2) = 0$.



(a) S_1 (b) S_2 (c) $S_1 \cup S_2$ (d) Decomposition of (c)

Fig. 3.12 Rigidity control by Theorem 4.

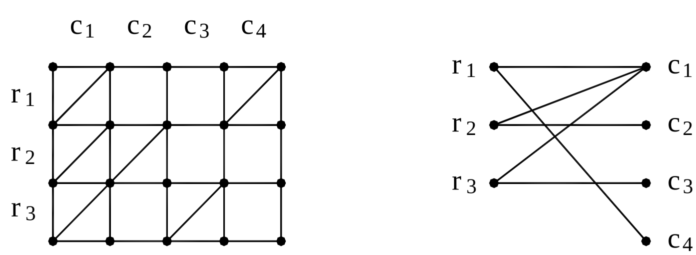
The above two examples suggest the use of a simple method for possible detection of the violation of the rigidity condition, which consists of decomposing S^{k+1} into its members and using Theorem 4. If a member has both ends in S^k, it should be added to S^k, since this does not increase $2N(S^k) - 3$, while decreasing the L.H.S. of Eq. (3-8) to become smaller than its R.H.S.

Although such a simple rule makes the quick detection of rigidity feasible, for a truss with a complex pattern one should employ a general method similar to those presented in Sections 3.3 and 3.4.

3.5.4 GRID-FORM TRUSSES WITH BRACINGS

Consider a planar truss consisting of square panels with or without diagonal members, an example of which is shown in Figure 3.13(a). The *bipartite graph* B(S) of S is a graph which is constructed as follows:

Associate one node with each row of squares and denote them by $r_1, r_2, \ldots, r_m$. With each column of squares, associate one node and denote them by $c_1, c_2, \ldots, c_n$ as depicted in Figure 3.13(b). Connect r_i to c_j if the corresponding squares in S have a diagonal member. The graph obtained in this manner is called the *bipartite graph* B(S) of S.



(a) (b)

Fig. 3.13 A planar truss and its bipartite graph B(S).

It is easy to prove that S is rigid, if and only if the corresponding bipartite graph B(S) is a connected graph. For this purpose, consider a square grid-form truss with two rows and two columns as depicted in Figure 14(a). A series of deformations can now be performed as shown in Figure 3.14(b).

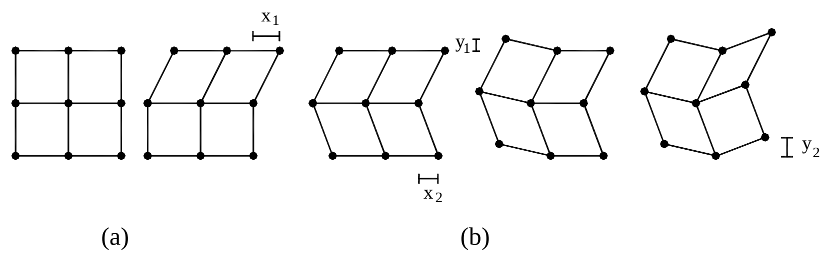


Fig. 3.14 A square grid-form truss and its deformation components.

Obviously, an arbitrary deformation of the truss can be considered as a combination of these deformation components. Now if a diagonal member is added to one of the squares, say the square corresponding to r_1 and c_1, then the deformation still takes place; however, a constraint in the form of $x_1=y_1$ is imposed. If a sufficient number of diagonal members are added, then $x_1=x_2=y_1=y_2$, and no square will deform relative to the squares, i.e. the entire truss will be rigid (if it is properly supported). This argument holds for any square grid form truss with m rows and n columns. Since the nodes of B(S) correspond to x_1, x_2, …x_m, y_1, y_2, …,y_m and the adjacency of r_1 and c_j in B(S) corresponds to the equality of $x_i=x_j$, therefore the result follows.

If B(S) is a spanning tree, then the corresponding S is generically independent and stiff. It can also be proven that the statical indeterminacy of S is the same as the first Betti number of B(S), Kaveh [92]. Figure 3.15(a) shows a rigid γ-tree for a planar truss, the corresponding bipartite graph of which is illustrated in Figure 3.15(b).

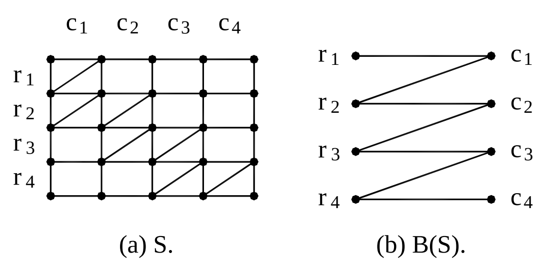


Fig. 3.15 A γ-tree and its tree bipartite graph.

3.6 Henneberg Sequence for Examining the Rigidity of Trusses

The following algorithm uses the Henneberg sequence to examine the rigidity of a given determinate planar truss. First it deletes all the nodes of degree 2 from the graph model of the truss, sequentially. Then a node i of degree 3, which is connected to nodes a, b and c, next, the incident members are omitted and a new member is added to the model between one of the pairs (a,b), (b,c) or (a,c), if that pair is not already connected in the previous model. Such a pair always exits; otherwise the truss would not have been a determinate structure. The above two operations are collectively called the *Henneberg sequence*. The following algorithm uses the Henneberg sequence until a single member is obtained, indicating the rigidity of the truss. The following theorem provides the proof for the existence of a node of degree 3 in a statically determinate planar truss:

Theorem [89]: For a statically determinate planar truss, there is at least one node of degree 2 or 3 for which Henneberg sequence can be applied.

Proof: Let the average degree of the nodes of S be defined as,

$$V_{ave} = \sum_{i+1}^{N(S)} \frac{V_i}{N(S)} = \frac{2M(S)}{N(S)},$$

where V_i is the degree of the node i. For a statically determinate planar truss, the degree of static indeterminacy $\gamma(S) = M(S) - 2N(S) + 3$ is equal to zero, i.e. $M(S) = 2N(S) - 3$, and:

$$V_{ave} = \frac{4N(S)-6}{N(S)} = 4 - \frac{6}{N(S)} < 4 .$$

Thus there should be a node j with $V_j = 2$ or 3 for which the Henneberg sequence can be applied.

Algorithm:

Step 1: Number the nodes and members of the graph model of the truss, separately.

Step 2: While there is a node of degree 2, delete such nodes and their incident members, one by one.

Step 3: If a node of degree 1 exists in S, go to Step 5.

Step 4: Delete a node of degree 3 with its incident members and connect two end nodes of the omitted members to make a non-repeated new member in S, and repeat steps 2-4. If such a node is not found, go to Step 6.

Step 5: If the remaining graph consists of a single member only, then the S is a rigid statically determinate truss. Otherwise it is not a determinate rigid truss.

Step 6: End.

Example: A statically determinate truss, shown in Figure 3.16 is examined for rigidity, using the above algorithm. Table 3.1 summarizes the operations performed in different steps of the Henneberg sequence algorithm. Member 9 remains at the end, indicating the rigidity of the truss.

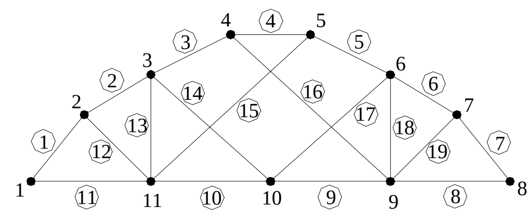


Fig. 3.16 A statically determinate planar truss.

It should be mentioned that the Henneberg sequence can also be applied in the form of an expansion, i.e. a node with two incident members can be added to S, and a node i with 3 incident members (i,a), (i,b) and (i,c) can also be replaced by a member (a,b), (a,c) or (b,c) of S.

The Henneberg sequence in the form of a mixed expansion and contraction (collapse), provides a suitable means to perform Müller-Breslau's bar exchange method, which is occasionally mistakenly associated with Henneberg in the literature. Such operations can also be used in the force method of structural analysis [89].

Table 3.1 Operations in the Henneberg sequence algorithm.

Step	Deleted node	Deleted members	New member
2	1	1,11	
2	2	2,12	
2	8	7,8	
2	7	6,19	
4	3	3,13,14	20(nodes 4,11)
4	4	4,16,20	21(nodes 5,9)
2	11	10,15	
2	5	5,21	
2	6	17,18	

3.7 CONNECTIVITY AND RIGIDITY

The *member-connectivity* $\lambda(S)$ of a connected graph S, is the smallest number of members the removal of which disconnects S. When $\lambda(S) \geq k$, then the graph S is called *k-member-connected*. A similar definition can be obtained for *node-connectivity* $\kappa(S)$ by replacing "members" with "nodes".

Attempts have been made to relate the connectivity of a graph to its rigidity. Some partial results have been obtained; however, no general approach is found for such an inter-relation. Some of the results obtained by Ref. [165] are outlined in the following:

Theorem 5: Every 6-connected graph is stiff.

As an example, the graph shown in Figure 3.17 is stiff; however, deleting four members makes it non-stiff.

It can be proven that a 5-connected graph is not necessarily stiff. As an example, for the graph of Figure 3.18, the rigidity can be checked as follows:

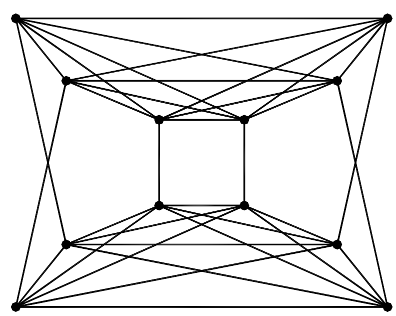


Fig. 3.17 A stiff 6-connected graph.

Decompose S into 8 complete subgraphs K_5 and leave the rest of the members as single subgraphs. The application of Eq. (3-8) results in,

$$8(2\times5 - 3) + 20(2\times2 - 3) \geq 2\times40 - 3, \text{ i.e. } 76 \geq 77,$$

which is false and therefore S in not rigid.

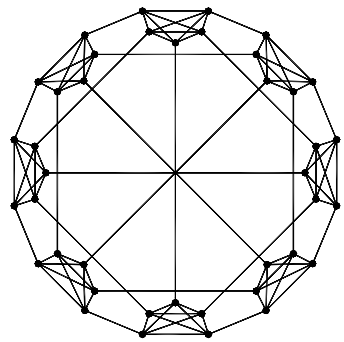


Fig. 3.18 A non-stiff 5-connected graph.

Finally it should be noted that many attempts have recently been made to extend these ideas to space trusses; however, thus far, no concrete result applicable to general space graphs has been obtained. Many open problems remain for further research, if pure graph-theoretical methods are to be developed for the recognition of the rigidity of space trusses. The theory of matroids, which is briefly introduced in Chapter 9 of this book, seems to be a promising tool for the study of rigidity.