CHAPTER 5

Matrix Displacement Method

5.1 INTRODUCTION

In the last half-century, considerable progress has been made in the matrix analysis of structures. The topic has been generalized to finite elements, and extended to the stability, non-linear and dynamic analysis of structures. This progress is due to the simplicity, modularity and flexibility of matrix methods.

Many textbooks covering these methods have been published including Argyris [4], McGuire and Gallagher [172], Livesley [160], Meek [173], Kardestuncer [88], ad Vanderbilt [242] among many others. In these books the displacement method of structural analysis is thoroughly developed, and therefore only a brief introduction will be presented here.

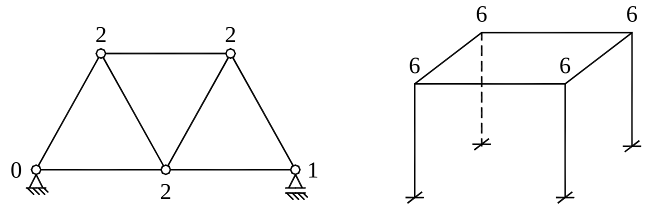
5.2 FORMULATION

In Chapter 4, the network-topological formulation of the displacement (stiffness) method of structural analysis has already been presented. In this section, a matrix formulation using the basic tools of structural analysis - equilibrium of forces, compatibility of displacements, and force-displacement relationships - is provided. The notations are chosen from the most popularly encountered versions in structural mechanics.

Consider a structure S with M members and N nodes; each node having one degree of freedom. The kinematical indeterminacy of S may then be determined as,

$$\eta(S) = \alpha N - \beta, \tag{5-1}$$

where β is the number of constraints due to the support conditions. As an example, $\eta(S)$ for the planar truss S depicted in Figure 5.1(a) is given by $\eta(S) = 2\times5 - 3 = 7$, and for the space frame shown in Figure 5.1(b), it is calculated as $\eta(S) = 6\times8 - 6\times4 = 24$.



(a) A planar truss. (b) A space frame.

Fig. 5.1 The degrees of freedom of the joints for two structures.

One can also calculate $\eta(S)$ by simple addition of the degrees of freedom of the joints of the structure, i.e. for the truss S, $\eta(S) = 2 + 2 + 2 + 1 = 7$, and for the frame $\eta(S) = 4\times6 = 24$.

Let **p** and **v** represent the joint loads and joint displacements of a structure. Then the force-displacement relationship for the structure can be expressed as,

$$\mathbf{p} = \mathbf{K}\mathbf{v}, \tag{5-2}$$

where **K** is a $\eta N\times\eta N$ symmetric matrix, known as the *stiffness matrix* of the structure. Expanding the *i*th equation of the above system, the force p_i can be expressed in terms of the displacements $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{\alpha N}\}$ as:

$$\mathbf{p}_i = \mathbf{K}_{i1}\mathbf{v}_1 + \mathbf{K}_{i2}\mathbf{v}_2 + ... + \mathbf{K}_{i\alpha N}\mathbf{v}_{\alpha N}. \tag{5-3}$$

A typical coefficient K_{ij} is the value of the force p_i required to be applied at the *i*th component of the structure, in order to produce a displacement $v_j=1$ at j and zero displacements at all the other components.

As has been shown in Chapter 4, the member forces **r** can be related to nodal forces **p** by:

$$\mathbf{p} = \mathbf{B}\mathbf{r}. \tag{5-4}$$

Similarly, the joint displacements **v** can be related to member distortions **u** by:

$$\mathbf{u} = \mathbf{B}^t\mathbf{v}. \tag{5-5}$$

For each individual member of the structure, the member forces can be related to member distortions by an element stiffness matrix $\mathbf{k}_m$. A block diagonal matrix containing these element stiffness matrices is known as the *unassembled stiffness matrix* of the structure, denoted by **k**. Obviously:

$$\mathbf{r} = \mathbf{k}\mathbf{u}. \tag{5-6}$$

This equation together with Eqs (5-4) and (5-5) yields:

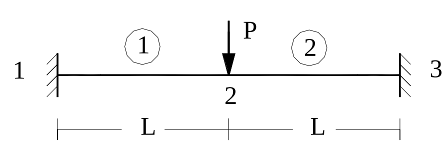
$$\mathbf{p} = \mathbf{B}\mathbf{k}\mathbf{B}^t\mathbf{v}. \tag{5-7}$$

Therefore,

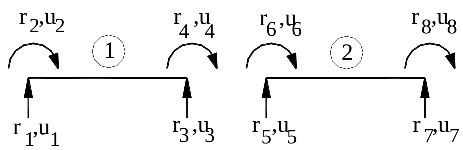
$$\mathbf{K} = \mathbf{B}\mathbf{k}\mathbf{B}^t \tag{5-8}$$

is obtained. The matrix **K** is singular since the boundary conditions of the structure are not yet applied. For an appropriately supported structure, the deletion of the rows and columns of **K** corresponding to the support constraints results in a positive definite matrix, known as the *reduced stiffness matrix* of the structure.

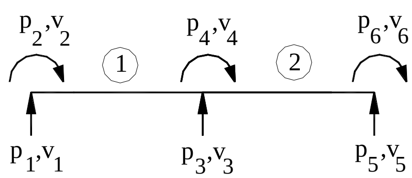
Let us illustrate the method by means of a simple example. Consider a fixed end beam with a load P applied at its mid span. This beam is discretized as two beam elements, as shown in Figure 5.2(a). The components of element forces and element distortions are depicted in Figure 5.2(b) and those of the entire structure are illustrated in Figure 5.2(c).



(a) A fixed ended beam S.



(b) Member forces and member distortions.



(c) Nodal forces and nodal displacements of the entire structure.

Fig. 5.2 Illustration of the analysis of simple structure.

For each element such as element 1, the element stiffness matrix can be written as,

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{5-9}$$

and for the entire structure we have:

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \tag{5-10}$$

Element stiffness matrices $\mathbf{k}_1$ and $\mathbf{k}_2$ can be easily constructed using the definition of k_{ij}. For a beam element, ignoring its axial deformation, these terms are shown in Figure 5.3. The structure has a uniform cross section and since both elements have the same length:

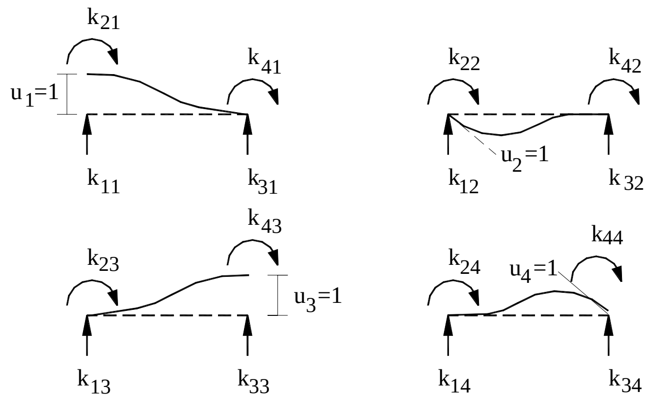


Fig. 5.3 The stiffness coefficients of a beam element ignoring its axial deformation.

$$\mathbf{k}_1=\mathbf{k}_2=\frac{2EI}{L}\begin{bmatrix}6/L^2 & -3/L & -6/L^2 & -3/L\\ -3/L & 2 & 3/L & 1\\ -6/L^2 & 3/L & 6/L^2 & 3/L\\ -3/L & 1 & 3/L & 2\end{bmatrix}. \tag{5-11}$$

The unassembled stiffness matrix is an 8×8 matrix of the form **k**:

$$\mathbf{k}=\begin{bmatrix}\mathbf{k}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{k}_2\end{bmatrix}. \tag{5-12}$$

Now consider the equilibrium of the joints of the structure, resulting in,

$$p_1=r_1\ ,\quad p_2=r_2\ ,\quad p_3=r_5+r_3,$$

$$p_4=r_4+r_6\ ,\quad p_5=r_7\ ,\quad p_6=r_8\ . \tag{5-13}$$

or in a matrix form we have,

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \\ r_6 \\ r_7 \\ r_8 \end{bmatrix}, \tag{5-14}$$

and more compactly:

$$\mathbf{p} = \mathbf{Br}. \tag{5-15}$$

Consider now the compatibility of displacements as:

$$u_1 = v_1 \;,\quad u_2 = v_2 \;,\quad u_3 = u_5 = v_3,$$

$$u_4 = u_6 = v_4 \;,\quad u_7 = v_5 \;,\quad u_8 = v_6 \,, \tag{5-16}$$

and in a matrix form we have,

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix}, \tag{5-17}$$

and in compact form:

$$\mathbf{u} = \mathbf{Ev} = \mathbf{B}^t\mathbf{v}. \tag{5-18}$$

The reason for matrix **E** being the transpose of the matrix **B**, has already been discussed in the previous chapter, however, using the principle of virtual work, a simple proof can be obtained. Consider:

$$\text{W} = \text{work done by external loads} = \frac{1}{2}\mathbf{v}^t\mathbf{p}$$

$$\text{U} = \text{strain energy} = \frac{1}{2}\mathbf{u}^t\mathbf{r}$$

Equating W and U leads to $\mathbf{E} = \mathbf{B}^t$ and completes the proof.

Therefore the stiffness matrix of the entire structure can be obtained as:

$$\mathbf{K} = \frac{2EI}{L}\begin{bmatrix} 6/L^2 & -3/L & -6/L^2 & -3/L & 0 & 0 \\ -3/L & 2 & 3/L & 1 & 0 & 0 \\ -6/L^2 & 3/L & 12/L^2 & 0 & -6/L^2 & -3/L \\ -3/L & 1 & 0 & 4 & 3/L & 1 \\ 0 & 0 & -6/L^2 & 3/L & 6/L^2 & 3/L \\ 0 & 0 & -3/L & 1 & 3/L & 2 \end{bmatrix}. \tag{5-19}$$

Applying the boundary conditions,

$$v_1 = v_2 = v_5 = v_6 = 0,$$

leads to the formation of the following reduced stiffness matrix:

$$\begin{bmatrix} p_3 \\ p_4 \end{bmatrix} = \frac{2EI}{L}\begin{bmatrix} 12/L^2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} v_3 \\ v_4 \end{bmatrix}. \tag{5-20}$$

Since $p_4 = 0$ and $p_3 = -P$, therefore $v_3 = \dfrac{p_3L^3}{24EI} = \dfrac{-PL^3}{24EI}$.

From this simple example, it can be seen that matrix **B** is a very sparse boolean matrix and the direct formation of $\mathbf{BkB}^t$ using matrix multiplication requires a considerable amount of storage. In the following, it is shown that one can form $\mathbf{BkB}^t$ with an assembling process (known also as *planting*), as follows:

Consider an element "a" of a structure, as shown in Figure 5.4, for which the element stiffness matrix can be written as,

$$\mathbf{k}_a = \begin{bmatrix} \mathbf{k}_{ii} & \mathbf{k}_{ij} \\ \mathbf{k}_{ji} & \mathbf{k}_{jj} \end{bmatrix}, \tag{5-21}$$

i and j are the two end nodes of member a. Multiplication $\mathbf{BkB}^t$ has the following effect on $\mathbf{k}_a$:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{k}_{ii} & \mathbf{k}_{ij} \\ \mathbf{k}_{ji} & \mathbf{k}_{jj} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \mathbf{k}_{ii} & 0 & \mathbf{k}_{ij} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{k}_{ji} & 0 & \mathbf{k}_{jj} & 0 & 0 \end{bmatrix} \tag{5-22}$$

$$= \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{k}_{ii} & 0 & \mathbf{k}_{ij} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{k}_{ji} & 0 & \mathbf{k}_{jj} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

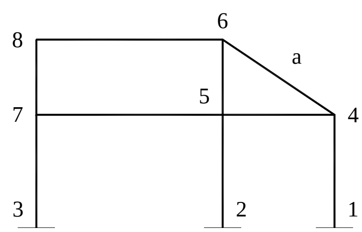


Fig. 5.4 A structural model S.

The adjacency matrix of S is also an 8×8 matrix and the effect of node 4 being adjacent to node 6, is the existence of unit entries in the same locations as the submatrices of the element a. One can build up the adjacency matrix of a graph by the addition of the effect of one member at a time. In the same way, one can also form the overall stiffness matrix of the structure by the addition of the contribution of every member in succession. As an example, for the graph shown in Figure 5.4, the overall stiffness matrix has the following pattern:

$$
\begin{array}{c c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{bmatrix}
* & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot \\
\cdot & * & \cdot & \cdot & * & \cdot & \cdot & \cdot \\
\cdot & \cdot & * & \cdot & \cdot & \cdot & * & \cdot \\
* & \cdot & \cdot & * & * & * & \cdot & \cdot \\
\cdot & * & \cdot & * & * & * & * & \cdot \\
\cdot & \cdot & \cdot & * & * & * & \cdot & * \\
\cdot & \cdot & * & \cdot & * & \cdot & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & * & * & *
\end{bmatrix}
\end{array}
. \tag{5-23}
$$

Non-zero entries are shown by $*$. For a stiffness matrix each of these non-zero entries is an $\alpha\times\alpha$ submatrix, where a is the degrees of freedom of each node of the structure. As an example, for a planar truss $\alpha = 2$, and for a space frame $\alpha = 6$. The formation of the stiffness matrix by the above process is known as *assembling* or *planting* of the stiffness matrix of a structure.

In the above example, the stiffness matrices could be assembled because both are constructed with reference to the same coordinate system. However, for a structure in general, the stiffness matrices should be prepared in a single coordinate system. On the other hand, for each element, there exists a coordinate system attached to the element, known as a *local coordinate system*. In Figure 5.5, local coordinate systems for members 45 and 25, and the global coordinate system for the entire structure are illustrated.

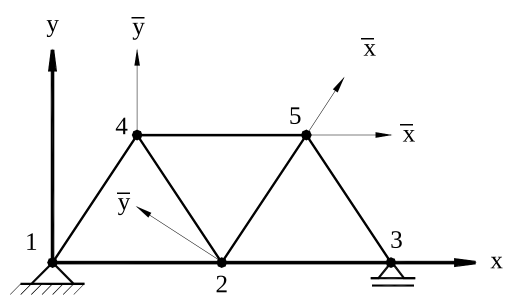


Fig. 5.5 Local $\bar{x}, \bar{y}$ and global coordinate x, y systems.

A global coordinate system can be selected arbitrarily; however, it may be advantageous to select this coordinate system such that the structure falls in the first quadrate of the plane, in order to have positive coordinates for the nodes of the structure. On the other hand, a local coordinate system of a member has one of its axes along the member, the second axis lies in its plane of symmetry (if it has one) and the third axis is chosen such that it results in a right handed coordinate system.

The transformation from a local coordinate to a global coordinate system can be performed as illustrated in Figure 5.6, in which xyz is the global system and $x_2y_2z_2$, often denoted by $\overline{xyz}$, is the local system.

The relation between $x_1y_1z_1$ and xyz can be expressed as:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{5-24}$$

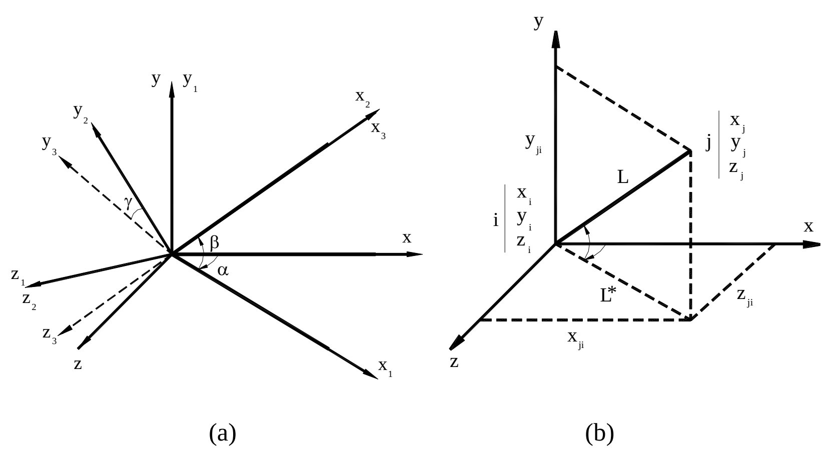


Fig. 5.6 Transformation from local coordinate system to global coordinate system.

Similarly $x_2y_2z_2$ and $x_1y_1z_1$ are related by,

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \tag{5-25}$$

and
$$\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & \sin\gamma \\ 0 & -\sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}. \tag{5-26}$$

Combining the above transformations results in,

$$\mathbf{T} = \begin{bmatrix} (\cos\alpha\cos\beta) & (\sin\beta) & (\cos\beta\sin\alpha) \\ -(\sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma) & (\cos\beta\cos\gamma) & (\sin\gamma\cos\alpha - \sin\alpha\sin\beta\cos\gamma) \\ -(\sin\alpha\cos\gamma - \cos\alpha\sin\beta\sin\gamma) & (-\cos\beta\sin\gamma) & (\cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma) \end{bmatrix}. \tag{5-27}$$

where:

$$\begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = [\mathbf{T}] \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{5-28}$$

A vector in a local coordinate system $\overline{\mathbf{\Gamma}}$ and in a global coordinate system $\mathbf{\Gamma}$ are related by:

$$\overline{\mathbf{\Gamma}} = \mathbf{T}\,\mathbf{\Gamma}. \tag{5-29}$$

It can easily be proved that **T** is an orthogonal matrix, i.e.

$$[\mathbf{T}]^{-1} = [\mathbf{T}]^{t}. \tag{5-30}$$

In the above transformation, γ represents the tilt of the member which is quite often zero. Thus **T** can be simplified as:

$$\mathbf{T} = \begin{bmatrix} \cos\alpha\cos\beta & \sin\beta & \sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & \cos\beta & -\sin\alpha\sin\beta \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}. \tag{5-31}$$

This matrix can easily be written in terms of the coordinates of the two ends of a vector. Considering Figure 5.6(b), Eq. (5-31) becomes,

$$\mathbf{T} = \begin{bmatrix} x_{ji}/L & y_{ji}/L & z_{ji}/L \\ -x_{ji}y_{ji}/L*L & L*/L & y_{ji}z_{ji}/L*L \\ -z_{ji}/L* & 0 & x_{ji}/L* \end{bmatrix}, \tag{5-32}$$

where:

$$x_{ji} = x_j - x_i \qquad y_{ji} = y_j - y_i \qquad z_{ji} = z_j - z_i$$

$$L^* = (z_{ji}^2 + x_{ji}^2)^{\frac{1}{2}} \text{ and } L = (z_{ji}^2 + y_{ji}^2 + x_{ji}^2)^{\frac{1}{2}}. \tag{5-33}$$

Notice that **T** transforms a 3-dimensional vector from a global to a local coordinate system and $\mathbf{T}^t$ performs the reverse transformation. However, if the element forces or element displacements (distortions) consist of p vectors, the block diagonal matrix with p submatrices should be used. As an example, for a beam element of a space frame, with each node having 6 degrees of freedom, the transformation matrix is a 12×12 matrix of the form:

$$\mathbf{T} = \begin{bmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \mathbf{T} & \\ & & & \mathbf{T} \end{bmatrix} \tag{5-34}$$

5.3 ELEMENT STIFFNESS MATRICES

Element stiffness matrices for skeletal structures can be obtained using various methods. For some elements, concepts from mechanics of solids are sufficient for the formation of an element stiffness matrix; for others, energy methods are more suitable. In the following, a general method for the formation of a stiffness matrix is presented and then applied to bar and beam elements. The details of the derivations are omitted for brevity. Such details can be found in any classical book on the matrix analysis of structures.

5.3.1 STIFFNESS MATRIX OF A GENERAL ELEMENT

Consider an elastic body as shown in Figure 5.7. Suppose that some loads are applied at certain points (specified as nodes) 1,2,...,n. Let v_{it} be the displacement of node i along the applied load p_{it}. The loads are applied in a pseudo-static manner, increasing gradually from zero. Assuming a linear behaviour, the work done by an external force $\mathbf{p} = \{p_1, p_2, \ldots, p_n)$ through the displacement $\mathbf{v} = \{v_1, v_2, \ldots, v_n\}$ can be written as:

$$W = \frac{1}{2}(p_1 v_1 + p_2 v_2 + \ldots + p_n v_n). \tag{5-35}$$

According to the principle of the conservation of energy,

$$W = U, \tag{5-36}$$

and therefore:

$$U = \frac{1}{2}(p_1 v_1 + p_2 v_2 + ... + p_n v_n) \; . \tag{5-37}$$

Now if a small variation is given to v_i while keeping the other displacement components constant, then the variation of **v** with respect to v_i can be written as:

$$\frac{\partial U}{\partial v_i} = \frac{1}{2}[p_i + \frac{\partial p_1}{\partial v_i} v_1 + \frac{\partial p_2}{\partial v_i} v_2 + ... + \frac{\partial p_n}{\partial v_i} v_n]. \tag{5-38}$$

According to Castigliano´s theorem:

$$\frac{\partial U}{\partial v_i} = p_i. \tag{5-39}$$

Thus,

$$p_i = [\frac{\partial p_1}{\partial v_i} v_1 + \frac{\partial p_2}{\partial v_i} v_2 + ... + \frac{\partial p_n}{\partial v_i} v_n], \tag{5-40}$$

or in a matrix form for all i=1,...,n we have:

$$\begin{bmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ p_n \end{bmatrix} = \begin{bmatrix} \frac{\partial p_1}{\partial v_1} & \frac{\partial p_2}{\partial v_1} & \cdot & \cdot & \frac{\partial p_n}{\partial v_1} \\ \frac{\partial p_1}{\partial v_2} & \frac{\partial p_2}{\partial v_2} & \cdot & \cdot & \frac{\partial p_n}{\partial v_2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial p_1}{\partial v_n} & \frac{\partial p_2}{\partial v_n} & \cdot & \cdot & \frac{\partial p_n}{\partial v_n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix}. \tag{5-41}$$

According to the definition, the above coefficient matrix forms the stiffness matrix of the elastic body defined by its n nodes as illustrated in Figure 5.7.

A typical element of the stiffness matrix k_{ij} is given by:

$$k_{ij} = \frac{\partial p_j}{\partial v_i}. \tag{5-42}$$

Using Castigliano´s first theorem:

$$k_{ij} = \frac{\partial}{\partial v_j}(\frac{\partial U}{\partial v_i}) = \frac{\partial^2 U}{\partial v_j \partial v_i}. \tag{5-43}$$

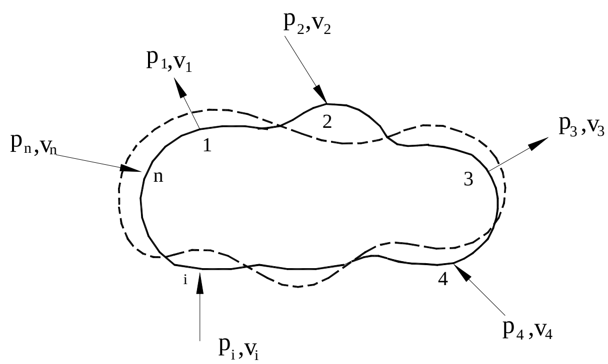


Fig. 5.7 An elastic body, its nodal forces and nodal displacements.

Similarly:

$$k_{ij} = \frac{\partial p_i}{\partial v_j} = \frac{\partial^2 U}{\partial v_j \partial v_i}. \tag{5-44}$$

Since the order of differentiation should not affect the result, we have,

$$k_{ij} = k_{ji}, \tag{5-45}$$

which is proof of the symmetry of the stiffness matrices, both for a structure and for an element.

A symmetric matrix **S** is called *positive definite*, if $\mathbf{x}^t\mathbf{S}\mathbf{x} > 0$ for every non-zero vector **x**. The stiffness matrix **K** of a structure is positive definite since,

$$\mathbf{p}^t\mathbf{v} = (\mathbf{K}\mathbf{v})^t\mathbf{v} = \mathbf{v}^t\mathbf{K}^t\mathbf{v} = \mathbf{v}^t\mathbf{K}\mathbf{v} = 2W,$$

and W is always positive.

5.3.2 STIFFNESS MATRIX OF A BAR ELEMENT

Consider a prismatic bar element as shown in its local coordinate system, Figure 5.8. According to the definition of such an element, only axial forces are present.

The strain energy of this bar can be calculated as:

$$U = \frac{1}{2}\iiint \sigma_{xx}\varepsilon_{xx}dxdydz = \frac{E}{2}\iiint \varepsilon_{xx}^2 dxdydz = \frac{EA}{2}\int \varepsilon_{xx}^2 dx \tag{5-46}$$

On the other hand:

$$\varepsilon_{xx} = \text{strain} = \frac{\partial u_x}{\partial x}. \tag{5-47}$$

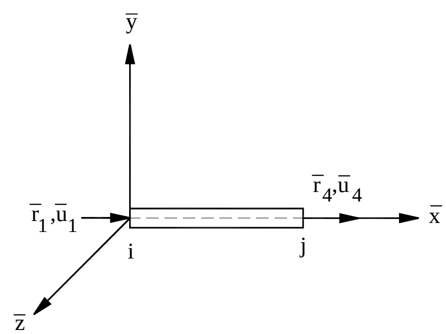


Fig. 5.8 A bar element in its local coordinate system.

Since the strain is constant along the bar, u_x can be expressed as:

$$u_x = A_1 x + A_2. \tag{5-48}$$

From the boundary conditions:

$$u_x = \bar{u}_1 \text{ at } x = 0$$

$$u_x = \bar{u}_4 \text{ at } x = L. \tag{5-49}$$

Hence: $$A_1 = \frac{\bar{u}_4 - \bar{u}_1}{L} \text{ and } A_2 = \bar{u}_1. \tag{5-50}$$

By substitution in Eq. (5-48), we have,

$$u_x = \frac{\bar{u}_4 - \bar{u}_1}{L} x + \bar{u}_1, \tag{5-51}$$

and from Eq. (5-46) the strain energy of the bar can be calculated as:

$$U = \frac{EA}{2L}[\bar{u}_4^2 - 2\bar{u}_4\bar{u}_1 + \bar{u}_1^2]. \tag{5-52}$$

Hence:

$$\bar{k}_{11} = \frac{\partial^2 U}{\partial \bar{u}_1^2} = \frac{EA}{L}, \quad \bar{k}_{14} = \bar{k}_{41} = \frac{\partial^2 U}{\partial \bar{u}_1 \partial \bar{u}_4} = -\frac{EA}{L}, \tag{5-53}$$

$$\bar{k}_{44} = \frac{\partial^2 U}{\partial \bar{u}_4^2} = \frac{EA}{L}, \text{ and } \bar{k}_{ij} = 0 \text{ for all other components.}$$

Therefore, the stiffness matrix of a bar element in the selected local coordinate system is obtained, and

$$\begin{bmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \bar{r}_3 \\ \bar{r}_4 \\ \bar{r}_5 \\ \bar{r}_6 \end{bmatrix} = \frac{EA}{L} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \bar{u}_4 \\ \bar{u}_5 \\ \bar{u}_6 \end{bmatrix} \tag{5-54}$$

From Eq. (5-29), we have:

$$\bar{\mathbf{r}} = \mathbf{Tr} \tag{5-55}$$

and

$$\bar{\mathbf{u}} = \mathbf{Tu} \tag{5-56}$$

From the definition of an element stiffness matrix in a local coordinate system:

$$\bar{\mathbf{r}} = \bar{\mathbf{k}}\bar{\mathbf{u}}\ . \tag{5-57}$$

By substitution of Eqs (5-55) and (5-56) in the above equation:

$$\mathbf{r} = \mathbf{T}^{-1}\bar{\mathbf{k}}\mathbf{Tu} = \mathbf{T}^{t}\bar{\mathbf{k}}\mathbf{T}\mathrm{u}. \tag{5-58}$$

By definition of a stiffness matrix in a global coordinate system:

$$\mathbf{r} = \mathbf{k}\mathbf{u}. \tag{5-59}$$

Comparison of Eq. (5-58) and Eq. (5-59) results in:

$$\mathbf{k} = \mathbf{T}^t\bar{\mathbf{k}}\mathbf{T}. \tag{5-60}$$

Hence the stiffness matrix of a bar element in a global system, as shown in Figure 5.9, can be written as:

$$\mathbf{k} = \begin{bmatrix} \mathbf{T} & \\ & \mathbf{T} \end{bmatrix}^t \left[\bar{\mathbf{k}}\right] \begin{bmatrix} \mathbf{T} & \\ & \mathbf{T} \end{bmatrix}.$$

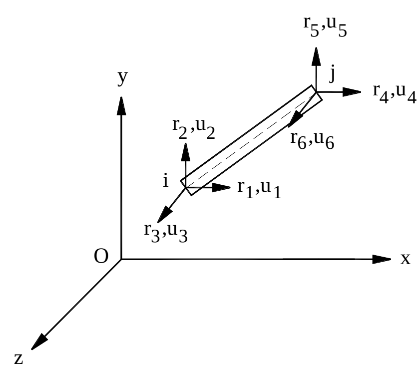


Fig. 5.9 A bar element of a space truss.

Denoting **T** in Eq. (5-32) by,

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}. \tag{5-61}$$

k can be written as:

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} T_{11}^2 & & & & & \\ T_{11}T_{12} & T_{12}^2 & & \text{sym.} & & \\ T_{11}T_{13} & T_{12}T_{13} & T_{13}^2 & & & \\ -T_{11}^2 & -T_{11}T_{12} & -T_{11}T_{13} & T_{11}^2 & & \\ -T_{11}T_{12} & -T_{12}^2 & -T_{12}T_{13} & T_{11}T_{12} & T_{12}^2 & \\ -T_{11}T_{13} & -T_{12}T_{13} & -T_{13}^2 & T_{11}T_{13} & T_{12}T_{13} & T_{13}^2 \end{bmatrix}. \quad (5\text{-}62)$$

The entries of the above matrix can be found using T_{ij} from Eq. (5.32). As an example, the stiffness matrix of bar 1 in the planar truss shown in Figure 5.10, can be obtained as:

$$T_{11} = \frac{x_{21}}{(x_{12}^2 + y_{12}^2 + z_{12}^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

$$T_{12} = \frac{y_{21}}{(x_{12}^2 + y_{12}^2 + z_{12}^2)^{\frac{1}{2}}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

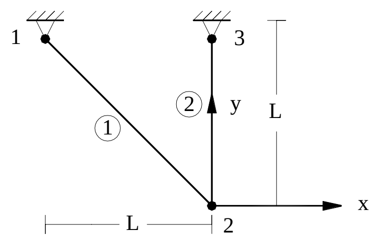


Fig. 5.10 A planar truss and the selected global coordinate system.

Therefore:

$$\mathbf{k}_1 = \frac{EA}{L\sqrt{2}}\left[\begin{array}{cc|cc} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ \hline -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{array}\right].$$

5.3.3 STIFFNESS MATRIX OF A BEAM ELEMENT

Consider a prismatic beam element as shown in Figure 5.11. The element forces and the element distortions, are defined by the following vectors:

$$\bar{\mathbf{r}} = \{r_1, r_2, r_3, \ldots, r_{12}\}^t,$$

and

$$\bar{\mathbf{u}} = \{u_1, u_2, u_3, \ldots, u_{12}\}^t,$$

where $r_1 - r_3$ are the force components at end i and $r_4 - r_6$ are moment components at end i. Also $r_7 - r_9$ are the force and $r_{10} - r_{12}$ are the moment components, respectively at the end j, and u_i (i=1,...,12) are correspondingly the translations and rotations at the ends i and j of the element.

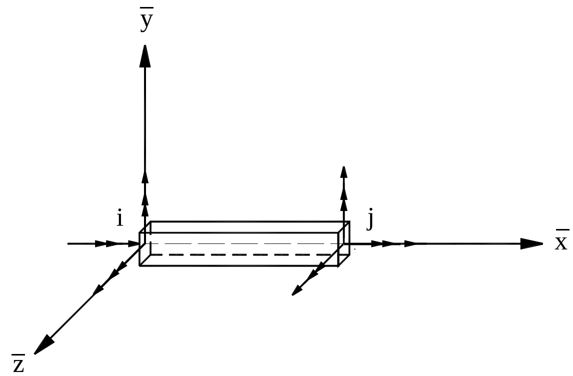


Fig. 5.11 A beam element in the local coordinate.

Using energy methods, the stiffness matrix of the beam element in the local coordinate system defined in Figure 5.11 can be obtained as:

$$\bar{\mathbf{k}} = \frac{E}{L}\begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 & -A & 0 & 0 & 0 & 0 & 0 \\ 0 & 12I_z/L^2 & 0 & 0 & 0 & 6I_z/L & 0 & 0 & -12I_z/L^2 & 0 & 0 & 6I_z/L \\ 0 & 0 & 12I_y/L^2 & 0 & -6I_y/L & 0 & 0 & 0 & -12I_y/L^2 & 0 & -6I_y/L & 0 \\ 0 & 0 & 0 & J/2(1+\nu) & 0 & 0 & 0 & 0 & 0 & -J/2(1+\nu) & 0 & 0 \\ 0 & 0 & -6I_y/L & 0 & 4I_y & 0 & 0 & 0 & -6I_y/L & 0 & 2I_y & 0 \\ 0 & 6I_z/L & 0 & 0 & 0 & 4I_z & 0 & -6I_z/L & 0 & 0 & 0 & 2I_z \\ -A & 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & 0 & 0 & 0 \\ 0 & -12I_z/L^2 & 0 & 0 & 0 & -6I_z/L & 0 & 12I_y/L^2 & 0 & 0 & 0 & -6I_z/L \\ 0 & 0 & -12I_y/L^2 & 0 & 6I_y/L & 0 & 0 & 0 & 12I_y/L^2 & 0 & 6I_y/L & 0 \\ 0 & 0 & 0 & -J/2(1+\nu) & 0 & 0 & 0 & 0 & 0 & J/2(1+\nu) & 0 & 0 \\ 0 & 0 & -6I_y/L & 0 & 2I_y & 0 & 0 & 0 & 6I_y/L & 0 & 4I_y & 0 \\ 0 & 6I_z/L & 0 & 0 & 0 & 2I_z & 0 & -6I_z/L & 0 & 0 & 0 & 4I_z \end{bmatrix} \tag{5-63}$$

In which I_y, I_z and J are the moments of inertia with respect to the $\overline{y}$ and $\overline{z}$ axes and J is the polar moment of inertia of the section. E specifies the elastic modulus and υ is the Poisson ratio. The length of the beam is denoted by L.

For the two-dimensional case the columns and rows corresponding to the third dimension can easily be deleted, to obtain the stiffness matrix of an element of a planar frame.

The stiffness matrix in a global coordinate system can be written as:

$$\mathbf{k} = \begin{bmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \mathbf{T} & \\ & & & \mathbf{T} \end{bmatrix}^t [\overline{\mathbf{k}}] \begin{bmatrix} \mathbf{T} & & & \\ & \mathbf{T} & & \\ & & \mathbf{T} & \\ & & & \mathbf{T} \end{bmatrix}. \tag{5-64}$$

For the two-dimensional case,

$$\mathbf{k} = \begin{bmatrix} \mathbf{T} & \\ & \mathbf{T} \end{bmatrix}^t [\overline{\mathbf{k}}] \begin{bmatrix} \mathbf{T} & \\ & \mathbf{T} \end{bmatrix}. \tag{5-65}$$

The entries of **k** are as follows,

$$k_{11} = T_{11}^2\alpha_1 + T_{21}^2\alpha_4^z$$

$$k_{21} = T_{11}T_{12}\alpha_1 + T_{21}T_{22}\alpha_4^z \qquad k_{22} = T_{12}^2\alpha_1 + T_{22}^2\alpha_4^z$$

$$k_{31} = T_{21}\alpha_2^z, \qquad k_{32} = T_{22}\alpha_2^z \qquad k_{33} = \alpha_3^z$$

$$k_{41} = -T_{11}^2\alpha_1 + T_{21}^2\alpha_4^z,\ k_{42} = -T_{21}T_{22}\alpha_4^z - T_{12}T_{11}\alpha_1,\ k_{43} = -T_{21}\alpha_2^z,$$
$$k_{44} = -T_{21}\alpha_2^z \tag{5-66}$$

$$k_{51} = -T_{21}T_{22}\alpha_4^z - T_{12}T_{11}\alpha_1 \qquad k_{52} = -T_{21}^2\alpha_4^z - T_{12}^2\alpha_1 \qquad k_{53} = -T_{22}\alpha_2^z$$

$$k_{54} = T_{21}T_{22}\alpha_4^z + T_{12}T_{11}\alpha_1 \qquad k_{55} = T_{22}^2\alpha_4^z + T_{12}^2\alpha_1$$

$$k_{61} = T_{21}\alpha_2^z,\ k_{62} = T_{22}\alpha_2^z,\ k_{63} = \alpha_6^z,\ k_{64} = -T_{21}\alpha_2^z,\ k_{65} = -T_{22}\alpha_2^z,\ k_{66} = \alpha_3^z.$$

in which:

$$\alpha_1 = \frac{EA}{L}, \quad \alpha_2^z = \frac{6EI_z}{L^2}, \quad \alpha_3^z = \frac{4EI_z}{L}, \quad \alpha_4^z = \frac{12EI_z}{L^3}, \text{ and } \alpha_6^z = \frac{2EI_z}{L}.$$

As an example, consider the planar frame, shown in Figure 5.12, with $A = 4\times10^{-3}\text{m}^2$, $I = 30\times10^{-6}\text{m}^4$ and $E = 2\times10^{11}\text{N/m}^2$. For element 1 we have,

$$T_{11} = 0 \qquad T_{12} = 1 \qquad T_{21} = -1 \qquad T_{22} = 0,$$

and the stiffness matrix of the element is obtained as,

$$\mathbf{k}_1 = 10^6 \left[\begin{array}{ccc|ccc} 1.25 & & & & & \\ 0 & 200 & & & \text{sym.} & \\ -0.75 & 0 & 6 & & & \\ \hline -1.25 & 0 & 0.75 & 1.25 & & \\ 0 & -200 & 0 & 0 & 200 & \\ -0.75 & 0 & 3 & 0.75 & 0 & 6 \end{array}\right],$$

where "sym." denotes the symmetry of the matrix.

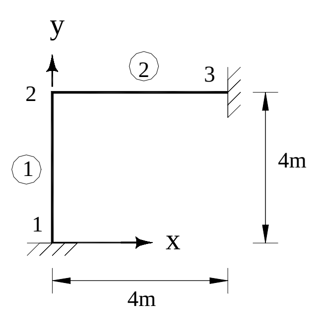


Fig. 5.12 A planar frame.

5.4 OVERALL STIFFNESS MATRIX OF A STRUCTURE

Once the stiffness matrix of an element is obtained in the selected global coordinate system, it can be planted in the specified and initialised overall stiffness matrix of the structure **K**, using the process described in Section 5.2. This is illustrated by the following simple example:

Let S be a planar truss with an arbitrary nodal and element numbering, as shown in Figure 5.13. The entries of the transformation matrices of the members are calculated using Eq. (5-32) and Eq. (5-33) as follows:

For bar 1: $T_{11} = \frac{x_2 - x_1}{2} = \frac{1-0}{2} = \frac{1}{2}$ and $T_{12} = \frac{y_2 - y_1}{2} = \frac{\sqrt{3}-0}{2} = \frac{\sqrt{3}}{2}$.

Similarly for bar 2: $T_{11} = \frac{1}{2}$ $T_{12} = -\frac{\sqrt{3}}{2}$, and for bar 3 $T_{11} = 1$, $T_{12} = 0$.

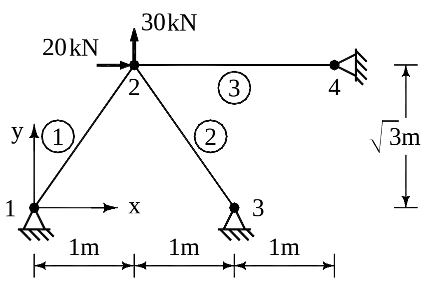


Fig. 5.13 A planar truss and the selected global coordinate system.

Now the stiffness matrices can be formed using Eq. (5-62) as:

For bar 1:
$$\mathbf{k}_1 = \frac{EA}{2}\begin{bmatrix} 0.25 & & & \text{sym.} \\ 0.433 & 0.75 & & \\ -0.25 & -0.433 & 0.25 & \\ -0.433 & -0.75 & 0.433 & 0.75 \end{bmatrix}.$$

For bar 2:
$$\mathbf{k}_2 = \frac{EA}{2}\begin{bmatrix} 0.25 & & & \text{sym.} \\ -0.433 & 0.75 & & \\ -0.25 & 0.433 & 0.25 & \\ 0.433 & -0.75 & -0.433 & 0.75 \end{bmatrix}.$$

For bar 3:

$$\mathbf{k}_3 = \frac{\text{EA}}{2}\begin{bmatrix} 1 & & & \text{sym.} \\ 0 & 0 & & \\ -1 & 0 & 1 & \\ 0 & 0 & 0 & 0. \end{bmatrix}.$$

The overall stiffness matrix of the structure is an 8×8 matrix, which can easily be formed by planting the three member stiffness matrices as follows:

$$\mathbf{K} = \frac{\text{EA}}{2}\begin{bmatrix} 0.25 & 0.433 & -0.25 & -0.433 & 0 & 0 & 0 & 0 \\ 0.433 & 0.75 & -0.433 & -0.75 & 0 & 0 & 0 & 0 \\ -0.25 & -0.433 & 1.5 & 0 & -0.25 & 0.433 & -1 & 0 \\ -0.433 & -0.75 & 0 & 1.5 & 0.433 & -0.75 & 0 & 0 \\ 0 & 0 & -0.25 & 0.433 & 0.25 & -0.433 & 0 & 0 \\ 0 & 0 & -0.433 & -0.75 & -0.433 & 0.75 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Partitioning **K** into 2×2 submatrices, it can easily be seen that it is pattern equivalent to the node adjacency matrix of the graph model of the structure as follows:

$$\mathbf{D}^* = \begin{bmatrix} * & * & \cdot & \cdot \\ * & * & * & * \\ \cdot & * & * & \cdot \\ \cdot & * & \cdot & * \end{bmatrix}.$$

This pattern equivalence simplifies certain problems in structural mechanics, such as ordering the variables for bandwidth or profile reduction, methods for increasing the sparsity using special cutset bases, and improving the conditioning of structural matrices, which will be discussed in Chapters 7 and 8.

The matrix **K** is singular, since the boundary conditions have to be applied. Consider,

$$\mathbf{p} = \mathbf{K}\mathbf{v}$$

and partition it for free and constraint degrees of freedom as:

$$\begin{bmatrix} \mathbf{p}_f \\ \mathbf{p}_c \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{ff} & \mathbf{K}_{fc} \\ \mathbf{K}_{cf} & \mathbf{K}_{cc} \end{bmatrix} \begin{bmatrix} \mathbf{v}_f \\ \mathbf{v}_c \end{bmatrix}. \tag{5-67}$$

This equation has a mixed nature; $\mathbf{p}_f$ and $\mathbf{v}_c$ have known values and $\mathbf{p}_c$ and $\mathbf{v}_f$ are unknowns. $\mathbf{K}_{ff}$ is known as the *reduced stiffness matrix* of the structure, which is non-singular for a rigid structure.

For boundary conditions such as $\mathbf{v}_c = \mathbf{0}$, it is easy to delete the corresponding rows and columns to obtain,

$$\mathbf{p}_f = \mathbf{K}_{ff}\mathbf{v}_f, \tag{5-68}$$

from which $\mathbf{v}_f$ can be obtained by solution of the above set of equations. In a computer this can be done by multiplying the diagonal entries of $\mathbf{K}_{cc}$ by a big number such as 10^{20}. An alternative approach is possible by equating the diagonal entries of $\mathbf{K}_{cc}$ to unity and all the other entries of these rows and columns to zero. If $\mathbf{v}_c$ contains some specified values, $\mathbf{p}_c$ will have corresponding $\mathbf{v}_c$ values. A third method, which is useful when a structure has more constraint degrees of freedom (such as many supports), consists of the formation of element stiffness matrices considering the corresponding constraints, i.e. to form the reduced stiffness matrices of the elements in place of their complete matrices. This leads to some reduction in storage, also at the expense of additional computational effort.

As an example, the reduced stiffness matrix of the structure shown in Figure 5.13 can be obtained from $\mathbf{K}$, by deleting the rows and columns corresponding to the three supports 1, 3 and 4.

$$\begin{bmatrix} 20 \\ 30 \end{bmatrix} = \frac{EA}{2} \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} u_{2x} \\ u_{2y} \end{bmatrix}.$$

The solution results in the joint displacements as:

$$u_{2x} = \frac{40}{1.5EA} \text{ and } u_{2y} = \frac{40}{EA}.$$

The member distortions can easily be extracted from the displacement vector, and multiplication by the stiffness matrix of each member results in its member forces in the global coordinate system. As an example, for member 3 we have:

$$\begin{bmatrix} r_{2x} \\ r_{2y} \\ r_{4x} \\ r_{4y} \end{bmatrix} = \frac{EA}{2} \begin{bmatrix} 1 & & & \\ 0 & 0 & \text{sym.} & \\ -1 & 0 & 1 & \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 40/1.5EA \\ 40/EA \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 13.33 \\ 0 \\ -13.33 \\ 0 \end{bmatrix}.$$

A transformation yields the member forces in the local coordinate systems, $\mathbf{r}_1 = \{-23.99 \;\; 23.99\}^t$, $\mathbf{r}_2 = \{-10.659 \;\; 10.65\}^t$ and $\mathbf{r}_3 = \{13.33 \;\; -13.33\}^t$.

Example: The truss shown in Figure 5.14 has members each of the same cross sectional area of 15000mm^2, and elastic modulus 210 kN/mm^2. Vertical loads of 10kN and 5kN are applied at node 3 and node 5, respectively. Determine the forces in all members:

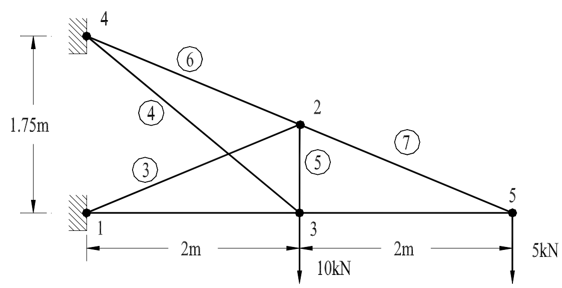


Fig. 5.14 A planar truss S.

The force-displacement relationship for a planar bar member is obtained from Eq. (5-23) as follows:

$$\begin{bmatrix} F_i^x \\ F_i^y \\ F_j^x \\ F_j^x \end{bmatrix} = \frac{EA}{L} \left[\begin{array}{cc|cc} T_{11}^2 & T_{11}T_{12} & -T_{11}^2 & -T_{11}T_{12} \\ T_{11}T_{12} & T_{12}^2 & -T_{11}T_{12} & -T_{12}^2 \\ \hline -T_{11}^2 & -T_{11}T_{12} & T_{11}^2 & T_{11}T_{12} \\ -T_{11}T_{12} & -T_{12}^2 & T_{11}T_{12} & T_{12}^2 \end{array}\right] \begin{bmatrix} \delta_i^x \\ \delta_i^y \\ \delta_j^x \\ \delta_j^y \end{bmatrix} \qquad (5\text{-}69)$$

The stiffness matrices for the members of S are determined as:

For members 1 and 2:

$$\begin{bmatrix} 1575.0 & 0 & -1575.0 & 0 \\ 0 & 0 & 0 & 0 \\ 1575.0 & 0 & -1575.0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1575.0 & 0 & -1575.0 & 0 \\ 0 & 0 & 0 & 0 \\ 1575.0 & 0 & -1575.0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For members 3 and 4:

$$\begin{bmatrix} 1210.71 & 530.02 & -1210.71 & -530.02 \\ 530.02 & 232.03 & -530.02 & -232.03 \\ -1210.71 & -530.02 & 1210.71 & 530.02 \\ -530.02 & -232.03 & 530.02 & 232.03 \end{bmatrix},$$

$$\begin{bmatrix} 672.08 & -588.18 & -672.08 & 588.18 \\ -588.18 & 514.76 & 588.18 & -514.76 \\ -672.08 & 588.18 & 762.08 & -588.18 \\ 588.18 & -514.76 & -588.18 & 514.76 \end{bmatrix}$$

For members 5 and 6:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3600.0 & 0 & -3600.0 \\ 0 & 0 & 0 & 0 \\ 0 & -3600.0 & 0 & 3600.0 \end{bmatrix}, \begin{bmatrix} 1210.71 & -530.02 & -1210.71 & 530.02 \\ -530.02 & 232.03 & 530.02 & -232.03 \\ -1210.71 & 530.02 & 1210.71 & -530.02 \\ 530.02 & -232.03 & -530.02 & 232.03 \end{bmatrix}$$

For member 7:

$$\begin{bmatrix} 1210.71 & -530.02 & -1210.71 & 530.02 \\ -530.02 & 232.03 & 530.02 & -232.03 \\ -1210.71 & 530.02 & 1210.71 & -530.02 \\ 530.02 & -232.03 & -530.02 & 232.03 \end{bmatrix}$$

Assembling the stiffness matrix of the entire structure and imposing the boundary conditions $\delta_1^x = \delta_1^y = \delta_4^x = \delta_4^y = 0$ results in:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ -10 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 3632.13 & & & & & \\ -530.02 & 4296.09 & & & \text{sym.} & \\ 0 & 0 & 3822.08 & & & \\ 0 & -3600.0 & -588.18 & 4114.76 & & \\ -1210.71 & 530.02 & -1575.0 & 0 & 2785.71 & \\ 530.02 & -232.03 & 0 & 0 & 530.02 & 232.03 \end{bmatrix} \begin{bmatrix} \delta_2^x \\ \delta_2^y \\ \delta_3^x \\ \delta_3^y \\ \delta_5^x \\ \delta_5^y \end{bmatrix}$$

The solution of the above equations results in the joint displacements:

$\delta_2^x = 4.716814 \times 10^{-3}$, $\delta_2^y = -2.12241 \times 10^{-2}$, $\delta_3^x = -1.09894 \times 10^{-2}$,
$\delta_3^y = -2.25665 \times 10^{-2}$ $\delta_5^x = -1.824108 \times 10^{-2}$ and $\delta_5^y = -9.521007 \times 10^{-2}$.

Once the displacements are calculated, the member forces can easily be obtained using member stiffness matrices.

5.5 GENERAL LOADING

The joint load vector of a structure can be computed in two parts. The first part comes from the external concentrated loads and/or moments, which are applied at the joints defined as the nodes of S. The components of such loads are most easily specified in a global coordinate system and can be entered to the joint load vector **p**.

The second part comes from the loads which are applied on members. These loads are usually defined in the local coordinate system of a member. For each member the fixed end actions (FEA) can be calculated using the existing classical formulae or tables. A simple computer program can be prepared for this purpose. The fixed end actions should then be rotated to the global coordinate system using the transformation matrix given by Eq. (5-27). The FEA should then be reversed and applied to the end nodes of the members. These components can be added to **p** to form the final joint load vector. After **p** has been prepared and the boundary conditions imposed, the corresponding equations should be solved to obtain the joint displacements of the structure. Member distortions can be extracted for each member in the reverse order to that used in assembling **p** vector.

Example: A portal frame is considered as shown in Figure 5. 15. The members are all made of sections with area A = 150cm^2, moment of inertia $I_z = 2\times10^4$cm^4 and elastic modulus E = 2×10^4kN/cm^2. Calculate the joint rotations and displacements.

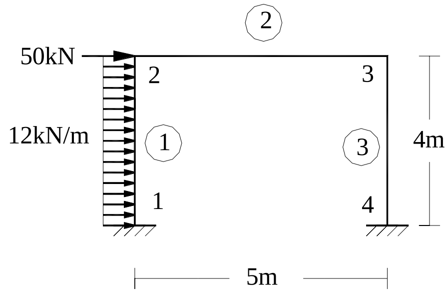


Fig. 5.15 A portal frame and its loading.

The equivalent joint loads are illustrated in Figure 5.16:

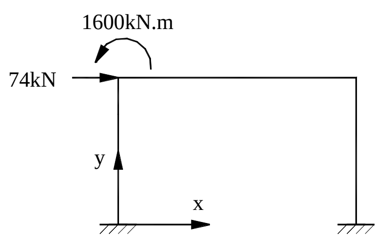


Fig. 5.16 Equivalent joint loads.

Employing Eq. (5-66), the stiffness matrices for the members are obtained as follows:

For member 1:

$$\mathbf{k}_1 = 10^4 \begin{bmatrix} 0.008 & & & & & \\ 0 & 0.75 & & & \text{sym.} & \\ -1.5 & 0 & 400 & & & \\ 0.008 & 0 & 1.5 & 0.008 & & \\ 0 & -0.75 & 0 & 0 & 0.75 & \\ -1.5 & 0 & 200 & 1.5 & 0 & 400 \end{bmatrix},$$

and for member 2:

$$\mathbf{k}_2 = 10^4 \begin{bmatrix} 0.6 & & & & & \\ 0 & 0.004 & & & \text{sym.} & \\ 0 & 0.96 & 320 & & & \\ -0.6 & 0 & 0 & 0.6 & & \\ 0 & -0.004 & -0.96 & 0 & 0.004 & \\ 0 & 0.96 & 160 & 0 & -0.96 & 320 \end{bmatrix}.$$

For member 3:

$$\mathbf{k}_3 = 10^4 \begin{bmatrix} 0.008 & & & & & \\ 0 & 0.75 & & & \text{sym.} & \\ 1.5 & 0 & 400 & & & \\ -0.008 & 0 & -1.5 & 0.008 & & \\ 0 & -0.75 & 0 & 0 & 0.75 & \\ 1.5 & 0 & 200 & -1.5 & 0 & 400 \end{bmatrix}.$$

Assembling the stiffness matrices and imposing the boundary conditions results in the following equations:

$$\begin{bmatrix} 7.4 \\ 0 \\ 160 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 10^4 \begin{bmatrix} 0.608 & & & & & \\ 0 & 0.754 & & & \text{sym.} & \\ 1.5 & 0.96 & 720 & & & \\ -0.6 & 0 & 0 & 0.608 & & \\ 0 & -0.004 & -0.96 & 0 & 0.754 & \\ 0 & 0.96 & 160 & 1.5 & -0.96 & 720 \end{bmatrix} \begin{bmatrix} \delta_2^x \\ \delta_2^y \\ \theta_2^z \\ \delta_3^x \\ \delta_3^y \\ \theta_3^z \end{bmatrix}$$

Solving these equations leads to:

$\delta_2^x = 0.0659167$, $\delta_2^y = 2.617764\text{E}{-04}$, $\theta_2^z = -8.983453\text{E}{-05}$,

$\delta_3^x = 0.06533767$, $\delta_3^y = -2.617704\text{E}{-04}$ and $\theta_3^z = -1.16855\text{E}{-04}$.

5.6 COMPUTATIONAL ASPECTS OF THE MATRIX DISPLACEMENT METHOD

The main advantage of the displacement method is its simplicity for computer programming. This is due to the existence of a simple kinematical basis formed on a special cutset basis known as cocycle basis of the graph model S of the structure. Such a basis does not correspond to the most sparse stiffness matrix, however, the sparsity is good enough, not to look for a better basis in more usual cases. However, if an optimal cutset basis of S is needed, then the displacement method has all the problems encountered in the force method, described in Chapter 6. The algorithm for the displacement method is summarized in the following. The coding for such an algorithm may be found in textbooks such as those of Vanderbilt [242] and Meek [173].

Algorithm

Step 1: Select a global coordinate system and number the nodes and members of the structure. An appropriate nodal ordering algorithm will be discussed in Chapter 7.

Step 2: After initialisation of all the vectors and matrices, read the data for the structure and its members. For multi-member regular structures, data can be generated using the method of Chapter 10.

Step 3: For each member of the structure:

(a) compute L, L*, sinα, sinβ,sinγ, cosα, cosβ, cosγ;

(b) compute the rotation matrix **T**;

(c) form the member stiffness matrix $\bar{\mathbf{k}}$ in its local coordinate system;

(d) form the member stiffness matrix **k** in the selected global coordinate system;

(e) plant **k** in the overall stiffness matrix **K** of the structure.

Step 4: For each loaded member:

(a) read the fixed end actions;

(b) transform the fixed end actions to the global coordinate system and reverse it to apply at joints;

(c) store these joint loads in the specified overall joint load vector.

Step 5: For each loaded joint:

(a) read the joint number and the applied joint loads;

(b) store it in the overall joint load vector.

Step 6: Apply boundary conditions to the structural stiffness matrix **K**, to obtain the reduced stiffness matrix $\mathbf{K}_{ff}$. Repeat the same for the overall joint load vector.

Step 7: Solve the corresponding equations to obtain the joint displacements.

Step 8: For each member:

(a) extract the member distortions from the joint displacements;

(b) rotate the member distortions to the local coordinate system;

(c) compute the member stiffness matrix;

(d) compute the member forces and fixed end actions.

Step 9: Compute the final member forces.

For an efficient displacement analysis of a structure, special considerations must be taken into account, which will be discussed in Chapters 7, 8 and 10 of this book.