CHAPTER 9

Matroids and Skeletal Structures

9.1 INTRODUCTION

The theory of matroids, which was introduced by Whitney [249] in his pioneering paper as early as 1935, is concerned with the abstract properties of independence. He conceived a "matroid" as an abstract generalization of a matrix; hence some of the language, including the name of the theory, is based on that of linear algebra. At the same time he refers to matroids as generalized graphs, and uses some terms from graph theory.

Matroids have received a great deal of attention from both the theoretical and the application points of view. Contributions have been made to its extension by Tutte, Rado, Welsh, Mirsky, Edmonds and many others; an excellent introduction is Welsh [243]. Matroids have been applied to various fields of engineering such as electrical networks by Minty [174], structural analysis by Kaveh [89,105] and rigidity of structures by Crapo [33] Recski [202], and Whiteley [248] among many other fields of science and engineering. The author´s interest in matroids has been motivated by the active presence of both matrices and graphs in the matrix analysis of structures, and the need for generalizing some of the existing concepts of graph theory, Refs. [93,94].

In this chapter, a matroid is defined using different but equivalent sets of axioms. Examples of matroids associated with graphs are included, due to the partial familiarity of structural analysts with the theory of graphs. The Greedy Algorithm developed by Edmonds [42] for selecting an optimal base of a matroid is described, and the problems encountered in the application of this algorithm in structural mechanics are discussed. Methods are suggested to overcome the difficulties involved.

9.2 AXIOM SYSTEMS FOR A MATROID

A matroid may be defined in many different inter-related forms, several of which were described in Whitney´s original paper. Here the definitions in terms of the concepts of independence, bases, circuits and ranks are presented, and the proof of their equivalence may be found in Refs. [243,249].

Matroid theory postulates certain sets to be independent and develops a fruitful theory from certain axioms which it requires to hold for this collection of independent sets.

9.2.1 DEFINITION IN TERMS OF INDEPENDENCE

A *matroid* $\mathcal{M}$ is a set of elements $\mathcal{S} = \{s_1, s_2, ..., s_m\}$ and a collection $\mathcal{F}$ of subsets of $\mathcal{S}$ (called independent sets) such that:

I1) $\varnothing \in \mathcal{F}$.

I2) If $X \in \mathcal{F}$ and $Y \subseteq X$, then $Y \in \mathcal{F}$.

I3) If $X \in \mathcal{F}$ and $Y \in \mathcal{F}$ with $| X | = | Y | + 1$, then there exists $s \in X - Y$ such that $Y+s \in \mathcal{F}$.

Here $| X |$ and $| Y |$ denote the cardinalities of the sets X and Y, respectively.

For a *matroid* $\mathcal{M} = (\mathcal{S},\mathcal{F})$, those subsets of $\mathcal{S}$ belonging to $\mathcal{F}$ are called *independent*, and those which do not belong to $\mathcal{F}$, are known as *dependent*. A maximal independent subset of a matroid, is known as a *base* of $\mathcal{M}$.

9.2.2 DEFINITION IN TERMS OF BASES

$\mathcal{M}(\mathcal{S},\mathcal{F})$ is a matroid if the collection of bases of $\mathcal{M}$, denoted by $\mathcal{B}$, satisfies the following conditions:

B1) $\mathcal{B} \neq \varnothing$.

B2) $| B_1 | = | B_2 |$ for every $B_1, B_2 \in \mathcal{B}$.

B3) If B_1, $B_2 \in \mathcal{B}$ and $s_1 \in B_1$, then there exists a $s_2 \in B_2$ such that $(B_1 - s_1 + s_2) \in \mathcal{B}$.

A *circuit* of a matroid $\mathcal{M}$ is a minimal dependent set of $\mathcal{S}$.

9.2.3 DEFINITION IN TERMS OF CIRCUITS

$\mathcal{M}(\mathcal{S},\mathcal{F})$ is a *matroid* if the collection of circuits of $\mathcal{M}$, denoted by $\mathcal{C}$, satisfies the following postulates:

C1) No proper subset of a circuit is a circuit.

C2) If C_1 and C_2 are distinct circuits of $\mathcal{C}$ and $s \in C_1 \cap C_2$, then there exists a circuit:

C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - s$.

Corresponding to each subset F_i, a number $r(F_i) \in Z$ is defined, which is known as the *rank* of F_i, as follows:

$$r(F_i) = \text{Max } \{|X|: X \subseteq F_i , X \in \mathcal{F} \}. \tag{9-1}$$

9.2.4 DEFINITION IN TERMS OF RANK

$\mathcal{M}(\mathcal{S},\mathcal{F})$ forms a *matroid* if the following conditions hold:

R1) The rank of the null subset is zero.

R2) For a subset F_i and any element s not in F_i, $r(F_i + s) = r(F_i) + k$, $(k = 0 \text{ or } 1)$.

R3) For s_1, s_2 not in F_i, if $r(F_i + s_1) = r(F_i + s_2) = r(F_i)$, then $r(F_i + s_1 + s_2) = r(F_i)$.

The *nullity* of F_i is defined as $n(F_i) = |F_i| - r(F_i)$. Obviously, a subset is independent if its nullity is zero; otherwise it is dependent. An element $s \in \mathcal{S}$ is called *dependent* on F_i if $r(F_i + s) = r(F_i)$; otherwise it is *independent* of F_i. The nullity of a base is zero, and that of a circuit is unity.

Now it is obvious that knowledge of the bases, or circuits, or rank functions is sufficient to uniquely determine the corresponding matroid. Therefore, it is not surprising that there exist axiom systems for a matroid in terms of each of these concepts. One can also consider one of these axiom systems and prove the others as theorems. A knowledge of all the axiom systems, helps in constructing suitable matroids and employing the relevant properties efficiently.

Example 1: Consider the following matrix:

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 \end{matrix}$$
$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

over the field $\mathbb{R}$ of real numbers. The column set {1,2,3,4,5} of **A** and its independent subsets form a matroid $\mathcal{M}(\mathbf{A})$. The set of independent subsets of this matroid is obtained as I = {∅, {1}, {2}, {4}, {5}, {1,2}, {2,4}, {2,5}, {4,5}}, and the set of its circuits is $\mathcal{C}$ = {{3}, {1,4}, {1,2,5}, {2,4,5}}.

Example 2: Let S be the graph as shown in Figure 9.1. Consider a matroid $\mathcal{M}$(S), formed on the members $\{m_1,m_2,m_3,m_4,m_5\}$ of S, with circuit set as:

$\mathcal{C} = \{\{m_3\},\{m_1,m_4\},\{m_1,m_2,m_5\},\{m_2,m_4,m_5\}\}$. This matroid is the known as the *cycle matroid* of the graph S, as defined in the Section 9.3.4.

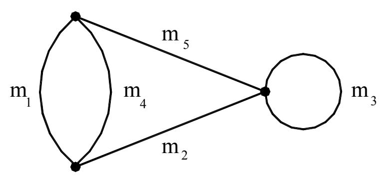


Fig. 9.1 A planar graph S.

Now compare $\mathcal{M}$(S) with $\mathcal{M}(\mathbf{A})$ in Example 1. It can be seen that, under bijection Ψ from {1,2,3,4,5} to $\{m_1,m_2,m_3,m_4,m_5\}$ defined by $\Psi(i) = m_i$, a set X is a circuit in $\mathcal{M}(\mathbf{A})$ if and only if Ψ(X) is a circuit in $\mathcal{M}$(S). Equivalently, a set Y is independent in $\mathcal{M}(\mathbf{A})$ if and only if Ψ(Y) is independent in $\mathcal{M}$(S). Thus matroid $\mathcal{M}(\mathbf{A})$ and $\mathcal{M}$(S) have the same structure or are *isomorphic*. A matroid that is isomorphic to the cycle matroid of a graph is called *graphic*. Therefore, the matroid $\mathcal{M}(\mathbf{A})$ in Example 1 is graphic.

9.3 MATROIDS RELEVANT TO STRUCTURAL MECHANICS

Matroids have been applied to various problems in structural analysis and the study of the rigidity of skeletal structures. In this section, examples of such matroids are considered, and the properties associated with each one are discussed.

9.3.1 A BASIS FOR A FINITE VECTOR SPACE

A conceptual study of structural analysis using vector spaces has been made by Maunder [169]. One can easily obtain a matroidal version of this study, by constructing matroids of the following kind:

Let $\mathcal{V}$ be a finite vector space and $\mathcal{F}$ be the collection of linearly independent subsets of vectors of $\mathcal{V}$. Then $\mathcal{M} = (\mathcal{V}, \mathcal{F})$ forms a matroid. The rank function of this matroid is the dimension of $\mathcal{V}$, and its base forms a basis of the vector space.

Although a finite vector space always constitutes a matroid, not all matroids are realizable as vector spaces.

9.3.2 A BASIS FOR CYCLE SPACE OF A GRAPH

A cycle basis of a graph is defined in Chapter 1, and its application for the formation of a statical basis of a structure is described in Chapter 6. In this section, a cycle space and its bases are formulated in terms of matroids.

Let $\mathcal{C}$ contain all simple cycles of a graph S, and $\mathcal{F}$ be the collection of mod 2 independent cycles of S. Then $(\mathcal{C},\mathcal{F})$ forms a matroid, defined as *cycle space matroid* $\mathcal{M}_S(S)$ of S. A base of $\mathcal{M}_S(S)$ is a cycle basis of S, and its rank is $M(S) - N(S) + b_0(S)$.

The above matroid can be defined using the member-cycle incidence matrix of a graph. Each row of this matrix corresponds to a member, and each column represents a cycle.

The columns of a member-cycle incidence matrix are either dependent or independent. Take the columns of the matrix as elements of $\mathcal{C}$, and independent subsets of columns as elements of $\mathcal{F}$. Then $(\mathcal{C},\mathcal{F})$ forms a cycle space matroid $\mathcal{M}_S(S)$ of S.

Example: Consider a graph S as shown in Figure 9.2. This graph contains 3 cycles $\mathcal{C} = \{C_1, C_2, C_3\}$ and $\mathcal{F} = \{(C_1),(C_2),(C_3),(C_1, C_2),(C_1, C_3),(C_2, C_3)\}$. The rank of $\mathcal{M}_S(S) = 5 - 4+1 = 2$, and $\{C_1, C_2\}$ is a typical base of this matroid.

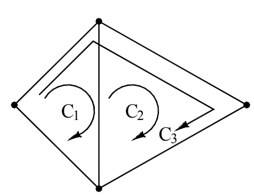


Fig. 9.2 A graph S and its cycles.

9.3.3 A BASIS FOR CUTSET SPACE OF A GRAPH

A cutset space of a graph is defined in Chapter 1, and its application for the formation of a kinematical basis of a structure, when the displacement method is used, is described in Chapter 5. In this section, a cutset space and its bases are defined in terms of matroids.

Let $\mathcal{C}^*$ contains all cutsets of a graph S, and $\mathcal{F}$ be the collection of mod 2 independent cutsets of S. Then, $(\mathcal{C}^*,\mathcal{F})$ forms a matroid, defined as *cutset space matroid* $\mathcal{M}_C$(S) of S. A base of $\mathcal{M}_C$(S) is a cutset basis of S, and its rank is given by N(S) − b_0(S). This matroid can also be defined using the member-cutset incidence matrix of S. The rows and columns of this matrix correspond to members and cutsets, respectively. The columns of this matrix are either dependent or independent. Take the columns of the matrix as elements of $\mathcal{C}^*$, and independent subsets of columns as elements of $\mathcal{F}$. Then $(\mathcal{C}^*,\mathcal{F})$ forms a cutset space matroid $\mathcal{M}_C$(S) of S.

Example: Consider a graph as shown in Figure 9.3. The non-empty elements of the set $\mathcal{C}^*$ contains 7 cutsets $\{C_1^*, C_2^*, C_3^*, C_4^*, C_5^*, C_6^*, C_7^*\}$ and $\mathcal{F} = \{(C_1^*),$ (C_2^*), (C_3^*), (C_4^*), (C_5^*), (C_6^*), (C_6^*), (C_1^*, C_2^*), (C_1^*, C_3^*), (C_2^*, C_3^*), (C_1^*, C_4^*), ... ,$(C_1^*, C_6^*, C_7^*)\}$. A typical base of the cutset space matroid, can be taken as B_1 = $\{C_1^*, C_2^*, C_4^*\}$.

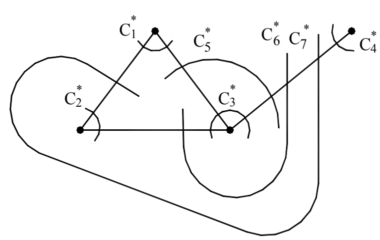


Fig. 9.3 A graph and its cutsets.

9.3.4 CYCLE MATROID OF A GRAPH

Spanning trees of a connected graph (spanning forest when S is not connected) have various applications. Some of its applications in structural engineering are described in Chapters 3, 5 and 6. In the following, a cycle matroid is defined in different inter-related forms, a base of which is a spanning tree of S:

Let S be a graph. Consider S as the set of members of S, and let $X \in \mathcal{F}$ if and only if X does not contain a cycle of S, i.e. it is a cycle-free subgraph (subtree if S is connected and subforest if it is disconnected). Then $\mathcal{F}$ is a collection of independent sets of a matroid in $\mathcal{S}$, known as the *cycle matroid* of S, denoted by $\mathcal{M}$(S). This matroid is called a *polygon matroid* by Tutte [241].

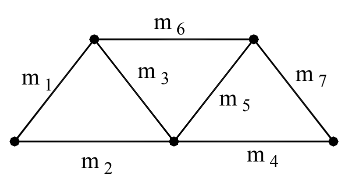
Alternatively, let S be a graph and consider the set of all spanning forests of S as $\mathcal{B}$. It can easily be shown that $\mathcal{B}$ is a base set of a matroid $\mathcal{M} = (\mathcal{S},\mathcal{F})$ on member set $\mathcal{M}$(S) of S, known as the *cycle matroid* of S.

Similarly, let C denote the set of simple cycles of a graph S, then C is the set of circuits of a matroid $\mathcal{M}$ on member set M(S), called a *cycle matroid* of S. The rank of $\mathcal{M}$(S) is $N(S) - b_0(S)$, and for a connected graph it is $N(S) - 1$.

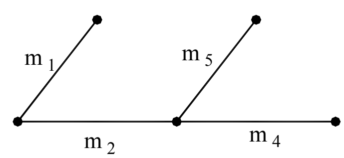
Example: Consider a graph S as shown in Figure 9.4(a). The sets $\mathcal{S}$ and $\mathcal{F}$ for the cycle matroid of S are as follows:

$\mathcal{S} = \{m_1,m_2,m_3,...,m_7\}$ and $\mathcal{F} = \{(m_1), (m_2), ...,(m_7), (m_1, m_2),(m_2, m_3), ..., (m_1, m_2, m_4), ...,(m_5,m_6,m_7), ..., (m_1,m_2,m_4,m_5)\}$.

A typical base of $\mathcal{M}$(S) may be considered as $B_1 = \{m_1,m_2,m_4,m_5\}$, and a typical circuit of $\mathcal{M}$(S) can be taken as $\{m_1,m_2,m_3\}$ or $\{m_1,m_2,m_5,m_6\}$.



(a) A graph S.



(b) A typical base of $\mathcal{M}$(S).

Fig. 9.4 A graph S and a typical base of its cycle matroid.

9.3.5 COCYCLE MATROID OF A GRAPH

Let S be a graph and $\mathcal{C}^*$ denote the set of cutsets of S. Then, $\mathcal{C}^*$ is the set of circuits of a matroid on $\mathcal{M}$(S), called a *cocycle* or *cutset matroid* of S, denoted by $\mathcal{M}^*$(S). Obviously, a set X of members of S is a base of the cocycle matroid $\mathcal{M}^*$(S), if and only if $M(S) - X$ is a spanning forest of S. For a connected graph, the members of $M(S) - T$ are known as cotrees of S. The rank of $\mathcal{M}^*$(S) is given as $r(\mathcal{M}^*(S)) = M(S) - N(S) + b_0(S)$.

Definition: Let $\mathcal{M}$ be a matroid on $\mathcal{S}$, whose bases are B_i. The collection of sets $\mathcal{S} - B_i$ are bases of another matroid $\mathcal{M}^*$ on $\mathcal{S}$, known as the *dual* matroid of $\mathcal{M}$. This dual matroid is unique for an $\mathcal{M}$, and the dual of a dual matroid is $\mathcal{M}$ itself. Circuits of $\mathcal{M}$ are called *cocircuits* or *cutsets* of $\mathcal{M}^*$.

By definition, it follows that the cycle matroid $\mathcal{M}(S)$ is the dual of the cocycle matroid $\mathcal{M}^*(S)$ of a graph S.

Example: Let S be a connected graph as shown in Figure 9.5.

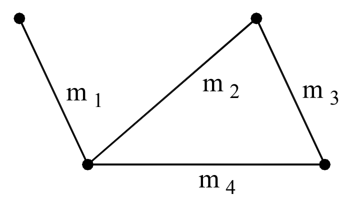


Fig. 9.5 A connected graph S.

The circuits of $\mathcal{M}(S)$ and $\mathcal{M}^*(S)$ are as follows:

$$\mathcal{C}(S) = \{m_2,m_3,m_4\},$$

and $$\mathcal{C}^*(S) = \{m_1\},\{m_2, m_4\},\{m_3, m_4\},\{m_2, m_3\}.$$

9.3.6 RIGIDITY MATROID OF A GRAPH

Graph theoretical approaches to the study of the rigidity of planar trusses are described in Chapter 3. Further investigation has proved the suitability of matroid theory to the study of rigidity, Refs [15,1210,249]. In this section, a simple definition of the rigidity matroid of a graph is given.

Let S be a graph, and define the *support* $\mathcal{N}(F_i)$ of a subset F_i of $\mathcal{M}(S)$ as the set of end nodes of members in F_i. Define a subset F_i of $\mathcal{M}(S)$ to be independent, if $|M(F_i')| \leq 2|N(F_i')| - 3$ holds for all subsets F_i' of F_i. These independent sets collectively form $\mathcal{F}$, and $(M(S),\mathcal{F})$ form a matroid known as a *two-dimensional generic rigidity matroid* $\mathcal{R}(S)$ of the graph S. A graph S is called *rigid*, if $r(\mathcal{M}(S)) = 2N(S) - 3$. A *circuit* of $\mathcal{R}(S)$ is a minimally dependent subset of $\mathcal{M}(S)$, examples of which are depicted in Figure 9.6.

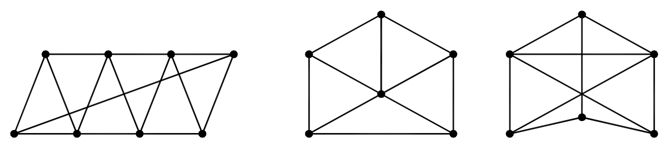


Fig. 9.6 Examples of circuits of $\mathcal{R}(S)$.

9.3.7 MATROID FOR NULL BASIS OF A MATRIX

Matroids are employed in combinatorial approaches to the force method of structural analysis, as will be described in Section 9.5. Matroids are also used in algebraic force methods, Ref. [74], a brief description of which is given here.

Let $\mathbf{Ax} = \mathbf{b}$ be the equilibrium equations of a structure, where $\mathbf{x}$ and $\mathbf{b}$ are the vectors of internal forces and applied loads. For a statically indeterminate structure, $\mathbf{A}$ is an m×n rectangular matrix with $n < m$ and rank $\mathbf{A} = m$.

Take the columns of $\mathbf{A}$ as the elements of $\mathcal{S}$ of a matroid $\mathcal{M} = (\mathcal{S},\mathcal{F})$ whose independent subsets are linearly independent subsets of the columns of $\mathbf{A}$. A circuit is a minimal dependent subset of columns. Generate all such circuits and consider it as $\mathcal{C} = \{C_1,C_2,...,C_r\}$. Now form another matroid $\mathcal{M}_n = \{\mathcal{C},\mathcal{F}_n\}$, where $\mathcal{F}_n$ consists of subsets of independent circuits of $\mathcal{C}$. A base of $\mathcal{M}_n$ is a null basis of $\mathbf{A}$, i.e. columns of a matrix $\mathbf{B}_1$ such that $\mathbf{AB}_1 = \mathbf{0}$.

For an efficient analysis, special null bases are required, which correspond to sparse flexibility matrices. The formation of such bases become feasible using the combinatorial optimisation method of Section 9.4.

9.4 COMBINATORIAL OPTIMISATION: THE GREEDY ALGORITHM

In 1926, Boruvka solved the following problem:

> Given a matrix of order n, having distinct positive real coefficients with
>
> $A_{ii} = 0$ and $A_{ij} = A_{ji}$, it is possible to find a set of coefficient such that:
>
> 1. There exist two randomly natural numbers k_1, k_2 ($\leq$ n) belonging to
>
>   the set of the form $A_{k1h2}, A_{h2h3},...,A_{hq\text{-}2hq\text{-}1}, A_{hq\text{-}1k2}$.

2. The sum of the terms of this set is minimal.

In graph theoretical terms, the above problem can be stated as follows:

For a connected graph with distinct positive real numbers assigned to its members, there is a shortest spanning tree, where the length of the tree is the sum of the numbers assigned to its branches.

After 30 years, Kruskal [146] in 1956 stated the above problem and gave three inter-related efficient algorithms for the selection of a shortest spanning tree of a connected graph. The uniqueness of the existence of such a tree was also proved in his paper. One of these methods is summarized in the following.

Let $\{m_i; i=1,2,...,M(S)\}$ be the member set of a graph S. Perform the expansion,

$$m_1 \rightarrow m_1 \cup m_2 \rightarrow \; ... \rightarrow T, \tag{9-2}$$

where m_{i+1} is chosen such that it has the smallest weight and does not form a cycle with $m_1 \cup m_2 \cup ... \cup m_i$. A shortest spanning tree will then be obtained.

This method formed a basis for the Greedy Algorithm for matroids, developed by Rado [199] and independently proved by three different authors, Refs. [42,54,243].

Greedy Algorithm

Let $\mathcal{M} = (\mathcal{S},\mathcal{F})$ be a matroid. Assign a positive value to each element of $\mathcal{S}$, denoted by W(s), $s \in \mathcal{S}$. For a subset $X \in \mathcal{F}$, define a weight function as,

$$W(X) = \Sigma W(s_i), \tag{9-3}$$

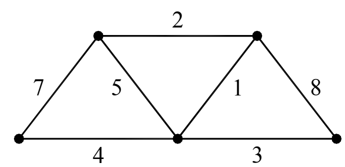
where summation is taken over all elements s_i of $X \subseteq \mathcal{S}$.

The problem is to find a subset X_{opt} of $\mathcal{S}$, such that $X_{opt} \in \mathcal{F}$ and $W(X_{opt})$ is minimum (or maximum) over all elements of $\mathcal{S}$. The Greedy Algorithm proceeds as follows:

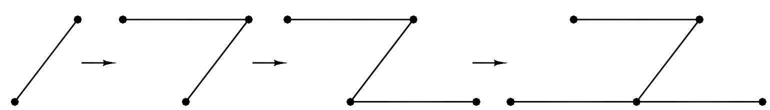
Select an element s_1 of minimal (maximal) measure (weight) from $\mathcal{S}$, and let $F_1 = \{s_1\}$. Form F_2 from F_1 by adding an element s_2 of minimal (maximal) measure such that F_2 is an independent set from $\mathcal{S} - \{s_1\}$, and let $F_2 = F_1+\{s_2\}$. Subsequently choose s_{i+1} of minimal (maximal) measure from $\mathcal{S} - \{s_1,s_2,...,s_i\}$ such that F_{i+1} is still an independent set. This process is clearly finite, and the finally selected set is a base of minimal (maximal) measure for $\mathcal{M}$.

An elegant proof of the minimality of the selected base may be found in Refs. [243,244].

Example: Consider a graph S as shown in Figure 9.7(a), with some positive weights assigned to its members. A base of minimal measure for cycle matroid $\mathcal{M}$(S), which is a spanning tree of minimal weight, is selected as depicted in Figure 9.7(b).



(a) A graph S.



(b) Expansion process of the Greedy Algorithm.

Fig. 9.7 A graph and the selected minimal base for its $\mathcal{M}$(S) matroid.

9.5 APPLICATION OF THE GREEDY ALGORITHM: A COMBINATORIAL FORCE METHOD

In Chapter 6 it is shown that, for an efficient force method, the sparsity of the flexibility matrix of a structure, which is pattern equivalent to the generalized cycle adjacency matrix of its graph model, should be maximized. This can be achieved by the use of a generalized cycle basis of minimal measure, where the weight of a γ-cycle is taken as its length (the number of its members). The Greedy Algorithm is a powerful means for this purpose. However, its application engenders certain difficulties, which form the remainder of this chapter.

Let a γ-cycle of a graph S be defined as a minimal subgraph C_i of S, which is rigid and $\gamma(C_i)$ = a. A maximal set of independent γ-cycles of S is known as a generalized cycle basis of S, the dimension of which is equal to $\eta(S) = \gamma(S)/a$. The integer "a" is defined in Table 2.1 (p. 44).

The set of all γ-cycles of S, together with $\mathcal{F}$ containing independent subsets of γ-cycles, form a matroid $\mathcal{M}_{gc}(S)$, called the *generalized cycle space matroid*. A base of this matroid is a generalized cycle basis of S. Therefore, the Greedy Algorithm selects a minimal generalized cycle basis of S.

Algorithm: Let the weight of a γ-cycle be measured by the number of its members. Select all γ-cycles of S, denoted by $\mathcal{C}$, and proceed as follows:

Step 1: Select the first γ-cycle of minimal length from $\mathcal{C}$.

Step 2: Take the second independent γ-cycle of minimal length from $\mathcal{C} - \{C_1\}$.

Step k: Subsequently choose a γ-cycle C_k of the least length from $\mathcal{C} - \{C_1, C_2, ..., C_{k-1}\}$ which is independent of the previously selected γ-cycles. Continue the process until β(S) of γ-cycles, forming a minimal generalized cycle basis, is generated.

Proof: Let $\overline{\beta}$ be an optimal generalized cycle basis of S, and let β be the basis selected by the Greedy Algorithm. Let the γ-cycles of β be ordered and denoted by:

$$C_k = C^k - C^{k-1} \text{ where } C^k = \bigcup_{i=1}^{k} C_i. \tag{9-4}$$

Then a γ-cycle $C_i \in \beta$ exists such that:

$$C_1, C_2, ..., C_{i-1} \in \beta \cap \overline{\beta} \qquad C_j \in \overline{\beta}. \tag{9-5}$$

By the well-known exchange theorem of matroids, there exists $\overline{C}_k \in \overline{\beta}$ such that,

$$\beta^* = \{ \overline{\beta} + C_j - \overline{C}_k \}, \tag{9-6}$$

forms a basis. Moreover, $\overline{C}_k \in C^{i-1}$ must hold or else β^* would contain only β(S) −1 distinct γ-cycles.

By definition, $L(\overline{\beta}) \le L(\beta^*)$, which implies $L(C_j) \ge L(\overline{C}_k)$. But the inequality cannot hold since $\overline{C}_k$ would then have been selected in place of C_j by the algorithm at that stage, and therefore $L(\beta^*) = L(\bar{\beta})$. Hence β^* is also an optimal basis which has more γ-cycles in common with β than $\bar{\beta}$. Continuing to reduce in this way, it becomes evident that $L(\beta) = L(\bar{\beta})$ and thus β forms a minimal generalized cycle basis for S.

9.6 PROBLEMS WITH APPLICATIONS OF THE GREEDY ALGORITHM

In practice, three main difficulties are encountered in an efficient use of the Greedy Algorithm:

1. The formation of a γ-cycle; provision of its rigidity.

2. The formation of all γ-cycles of S.

3. Checking the independence of each selected γ-cycle.

In the following, the above problems are discussed for different types of skeletal structures, listed in Table 2.1:

9.6.1 PLANAR AND SPACE FRAMES

For this type of structure, $\gamma(S) = \alpha b_1(S)$ (α=3 or 6), and α S.E.Ss can be generated on each cycle of the selected cycle basis. For maximal sparsity of flexibility matrices, optimal cycle bases should be formed. However, minimal cycle bases are often preferred due to simplicity and applicability of the Greedy Algorithm and many other topological methods. Only minimal cycle bases are discussed here.

First Problem: For this type of skeletal structures, a cycle is rigid both in the plane and in the space; therefore rigidity is no problem.

Second Problem: Formation of all simple cycles of a graph, especially for large structures, is quite time-consuming and impractical. Horton [82] limited the number of cycles to be formed in conjunction with the Greedy Algorithm to $M(S)\times M(S)$. The author´s expansion process requires far fewer cycles to be generated, of course at the expense of the formation of slightly longer cycle bases.

Third Problem: The simplest method for ensuring independence is the use of the chords of a spanning tree as the generators of independent cycles. This, in general, leads to the formation of very long cycles. This algorithm is modified by employing an SRT rooted at an appropriate starting node. Further improvement has been achieved by using ordered chords and addition of the used chord to tree branches at each stage of the expansion. Independence can also be ensured using the admissibility condition, restricting the increase of cycle rank by unity in each step of an expansion process, as has been described in Chapter 6. An alternative method is the use of a completion process, Ref. [83]. A simple but rather expensive approach to ensure independence is the application of an algebraic method such as Gaussian elimination. However, this approach reduces the order dependency of the expansion process. A mixed version of admissibility condition and Gaussian elimination is recommended for the generation of shorter cycle bases.

It should be added that, future research in this area should be directed to the formation of optimal and optimally conditioned cycle bases.

9.6.2 PLANAR TRUSSES

For this type of structure, $\gamma(S) = M(S) - 2N(S) + 3$, and one S.E.S. is generated on each γ-cycle of S. For a general planar truss, a generalized cycle basis may be used for the formation of a suitable statical basis, which involves the following problems:

First Problem: The rigidity of a γ-cycle (or a γ-tree when a fundamental generalized cycle basis is used) should be provided by one of the methods presented in Chapter 3. The matching algorithm of Sugihara [227] is more suitable for this purpose.

Second Problem: Obviously, the formation of all rigid γ-cycles for a multi-member truss is laborious and time-consuming. Therefore an expansion process, with the selection of one γ-cycle of minimal length at each step, seems to be a reasonable approach to overcome this difficulty.

Third Problem: For checking the independence of the γ-cycles, either a rigid γ-tree should be used, or the admissibility condition of Chapter 6 should be employed. An alternative approach is to use Gaussian elimination, which requires a considerable amount of storage.

9.6.3 SPACE TRUSSES

For this type of skeletal structure, γ(S) = M(S) − 3N(S) + 6, and one S.E.S. should be formed on each γ-cycle.

First Problem: Although some studies have been carried out for the rigidity of space trusses, further research is needed for efficient implementation of the developed ideas. These studies are confined to specific configurations, and it is believed at this stage that algebraic approaches, or mixed algebraic-topological methods, are more suitable for this type of skeletal structure, Refs. [58,87,238].

Second Problem: Formation of all γ-cycles for space structures having a considerable number of members is very difficult, and again an expansion process, with the selection of one γ-cycle of smallest length at a time, seems to be the only practical approach, once the problem of the rigidity is solved.

Third Problem: As for planar trusses, this problem does not produce a serious difficulty in the process of selection of a generalized cycle basis.

9.7 FORMATION OF SPARSE NULL BASES

9.7.1 DEFINITIONS

The bipartite graph B(**A**) of a matrix **A** can be constructed by associating one row-node with each row i and one column-node with each column j if the corresponding entry a_{ij} of **A** is non-zero. As an example, the matrix **A** and the corresponding bipartite graph B(**A**), together with a matching in **A** and its graph, are shown in Figure 9.8:

$$\mathbf{A} = \begin{bmatrix} \otimes & \times & 0 & 0 & \times \\ 0 & 0 & \otimes & 0 & \times \\ \times & \otimes & \times & \times & 0 \end{bmatrix}$$

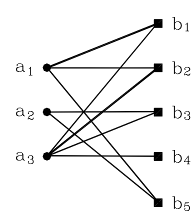


Fig. 9.8 A rectangular matrix **A** and its bipartite graph B(**A**).

It can be proved that a matrix **A** has a complete matching if and only if it has the Hall property; i.e. if subsets of its rows have non-zeros in at least as many columns. It has also been proved that the matching number of a matrix is greater than or equal to its rank. Therefore, a matrix with full rank has a complete matching.

9.7.2 NULL BASES FORMATION

A complete matching of B(**A**) partitions the columns of **A** into the set of matched columns M and a set of unmatched columns U. As an example in Figure 9.8, $\{b_1,b_2,b_3\}$ are matched and $\{b_4,b_5\}$ are unmatched columns. It is shown that, for a column $u \in U$, a circuit can be constructed using an alternating path algorithm (see Chapter 1).

An M-alternating path is a path whose members are alternately from the matching M and outside M. As an example, in Figure 9.8, $\{b_5,a_1,b_1,a_3,b_2\}$ is an M-alternating path in **A**. We say b_5 is *reachable* from b_1 and b_3, and show it with $b_5 \rightarrow b_1$ and $b_5 \rightarrow b_2$. An augmenting path is an alternating path, which begins and ends with unmatched nodes. The size of corresponding matching can be increased by making the members in the matching unmatched and vice versa.

For a member $u \in U$, a dependent set n(u) containing u can be considered, which is a circuit if **A** has the *Weak Haar Property* (WHP). A matrix has WHP if every set of columns C satisfies rank(C) = Ψ(C), where Ψ(C) is the matching number. This property ensures that n(u) will be a circuit for all numeric values of the columns of **A**. For a particular set of numeric values of the non-zero entries of **A**, numerical cancellation may occur, in which case the set n(u) will contain a circuit.

Therefore, for finding a circuit of a matrix with WHP, a complete matching M should be constructed and an unmatched column u should be selected. A circuit n(u) is formed by following all M-alternating paths from u and adding columns visited to n(u), i.e. $n(u) = u+\{v \in M: u \rightarrow v\}$. As an example, two circuits $n(b_4) = \{b_4,b_2,b_1\}$ and $n(b_5) = \{b_5,b_3,b_2,b_1\}$ can easily be formed. However, if n(u) does not have WHP, then it contains a circuit which should be identified by numerical factorization.

Once the formation of a circuit becomes possible, different algorithms can be designed for the formation of a null basis. Two such algorithms are given by Coleman and Pothen [29,30], and in the following an algorithm for the formation of a fundamental null basis is briefly discussed.

Let N be the empty set. Find a complete matching M of **A**, partitioning the columns of **A** as **A** = [**M U**]. Then, for each $u \in U$, construct a circuit n(u). Augment the null basis N with the computed null vector. This process should be repeated for all members of U in order to obtain a fundamental null basis of **A** with WHP. When **A** does not have WHP, then a fundamental basis can be computed only when M has full rank. Therefore, one should choose M by a matching, but should ensure that M has full rank while factoring it to compute the null vectors. When it is rank deficient, the dependent columns in M should be rejected, and a new maximum matching should be found. This will ensure the formation of a basis and will always succeed when **A** has full row rank.

Remarks: Matroids make the compact formulation of many structural problems feasible. Once a matroid has been constructed, the Greedy Algorithm can be used for the formation of its minimal base. However, the efficient application of the Greedy Algorithm requires special considerations, similar to those discussed in Section 9.6.