Chapter 5:

Non-conventional (Complex) Structures

Introduction

The grillages

The vierendeel girders

The arches

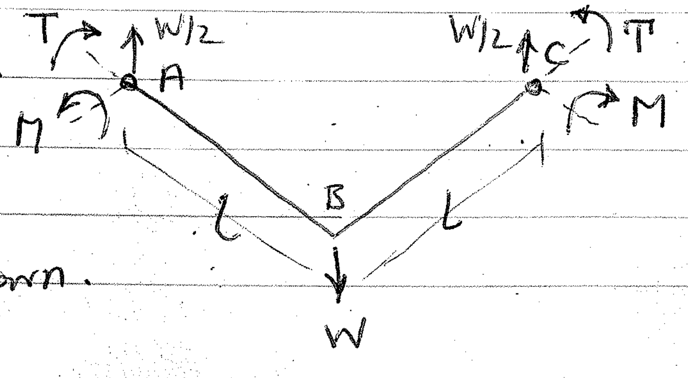
5
EXAMPLES OF COMPLEX FRAMES

Plastic analysis is also applicable to other structures such as grillages and space frames. The same limitations as plasic theory apply, i.e deflexions are assumed to be small and instability is supposed to be prevented.

Theory becomes insufficient for the full exploration of certain types of structures, which are discussed in volume 2.

5.1 Grillage

Straight members lying in the same plane, loads applied in perpendicular to plane. Where the members cross they are joined with full strength connexions so that any required forces and moments can be transmitted through the grillage.



Consider a simple space frame as shown.

This frame is built in at A and C. Collapse mechanism will be formed by the formation of plastic hinges at the ends A and C.

We have a combination of bending and Twisting that makes the plastic analysis of this structure difficult.

From Static

$$\frac{1}{2} WL = T + M. \qquad (5\text{-}1)$$

Another relation can be established between M & T as

$$\left(\frac{M}{M_p}\right)^2 + \left(\frac{T}{T_p}\right)^2 = 1 \qquad (5\text{-}2)$$

M_p = Full plastic Bending moment in the absence of twisting moment

T_p = " " Torque " " " " " Bending moment

i.e. Full plastic torque.

Although the mech. condition is satisfied by the arrangement of hinges, however, not enough information for the solution.

'T' can be eliminated between (5-1) & (5-2)

$$\frac{1}{2} Wl = M + \frac{T_p}{M_p}\sqrt{M_p^2 - M^2} \qquad (5\text{-}3)$$

Combined Bending and Torsion

For members with constant tube sections
The condition for yield can be written as

$$\left(\frac{M}{M_p}\right)^2 + \left(\frac{T}{T_p}\right)^2 = 1$$

proof: Consider a member under a combined action of bending moment M and torque T.

For a tube member with thin walled cross-section, with mean diameter d and thickness t have the following properties:

Under T, the shear stress can be calculated as τ by $\quad T = \frac{1}{2}\pi t d^2 \tau$

If the material follows Von Mises criteria, then σ is combined with τ as

$$\sigma^2 + 3\tau^2 = \sigma_0^2$$

σ_0 is the yield stress in simple tension.

The presence of T and hence τ reduces σ_0 to σ.

From chapter 1, for a thin-walled

circular cross-section we have

$$M = t d^2 \sigma$$

using the above equations yield

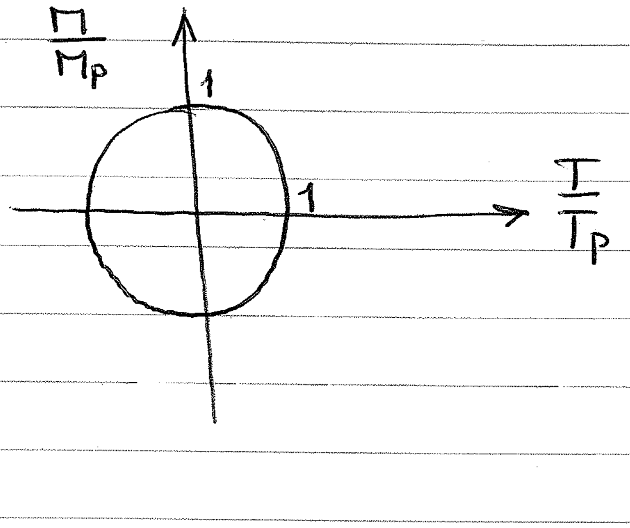
$$\left(\frac{M}{td^2}\right)^2 + 3\left(\frac{T}{(\frac{1}{2})\pi t d^2}\right)^2 = \sigma_0^2$$

But $M_p = t d^2 \sigma_0$ and $T_p = \frac{\pi}{2} t d^2 \frac{\sigma_0}{\sqrt{3}}$

Therefore

$$\left(\frac{M}{M_p}\right)^2 + \left(\frac{T}{T_p}\right)^2 = 1$$

which is plotted in $\frac{M}{M_p}$, $\frac{T}{T_p}$ coordinate system as a circle of unit length.



M is not yet determined. Values of M and T at a hinge at A and C must be adjusted so that the plastic work done in any motion of the collapse mech. is a maximum.

Thus in Eq (5-1), M should be chosen to make W a maximum, that is $\frac{dW}{dM} = 0$ or

$$1 - \frac{T_p}{M_p}\left(M_p^2 - M^2\right)^{-\frac{1}{2}} \cdot M = 0$$

or
$$M = \frac{M_p^2}{\sqrt{M_p^2 + T_p^2}} \qquad (5\text{-}4)$$

Substitution in (5-3) leads to

$$\frac{1}{2} W_c l = \sqrt{M_p^2 + T_p^2} \qquad (5\text{-}5)$$

This condition relates possible deformations at hinges A and C directly to yield equation (5-2).

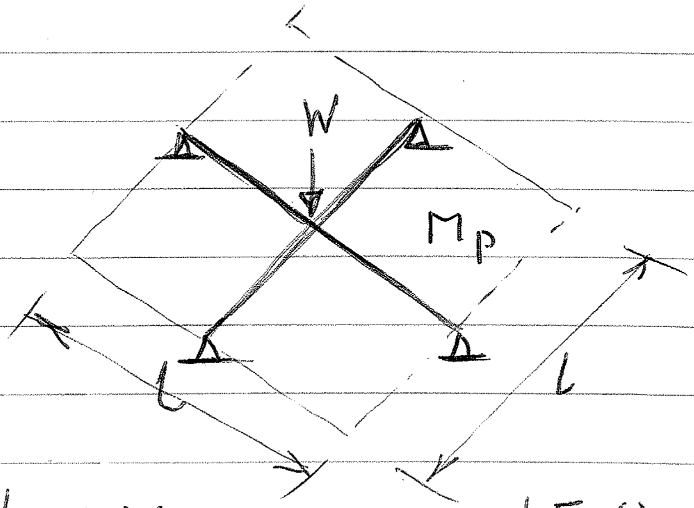
Even such a simple problem raises difficulties if an exact solution is sought. However, for I-sections the strength in twisting is small, and T_p can be neglected compared to M_p.

Therefore $\frac{1}{2} W_c l = M_p$.

This is equivalent to ignoring T in figure (5-1), and Eq. (5-2) becomes $M^2 = M_p^2$.

Indeed it is only for tubular and box sections appropriate twisting moment can be developed. Further examples given here will assume that all twisting effects are negligible.

Example 1:



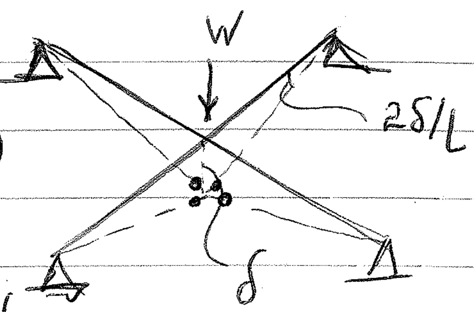
Clearly two beams act in series, and the collapse load is given by

$$\frac{1}{4}WL = 2M_p \qquad (5\text{-}6)$$

This equation can be derived by writing work balance for a small movement δ of the load.

$$W\delta = \left[M_p\left(\frac{4\delta}{L}\right)\right] + \left[M_p\left(\frac{4\delta}{L}\right)\right] \qquad (5\text{-}7)$$

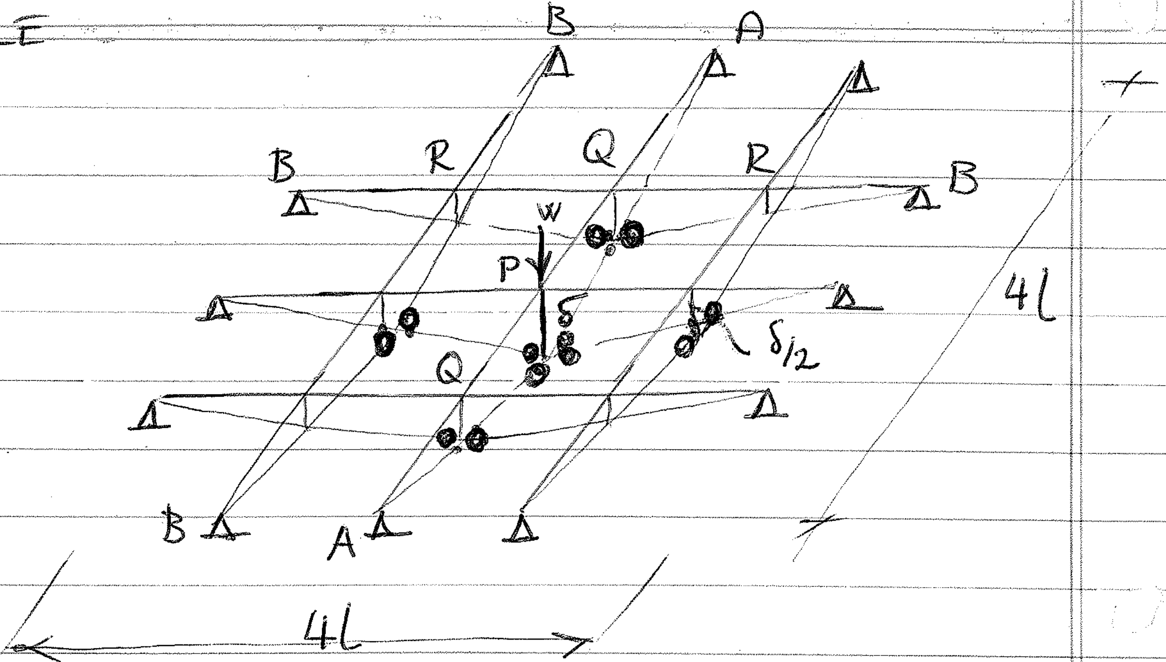
or $$W = \frac{8M_p}{L}$$



If full plastic moments for beams are different as M_1 and M_2, then

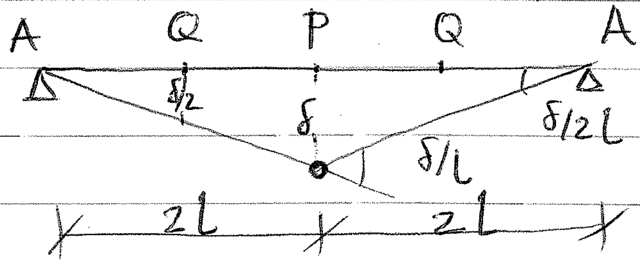
$$W = (M_1 + M_2)(4/L) \qquad (5\text{-}8)$$

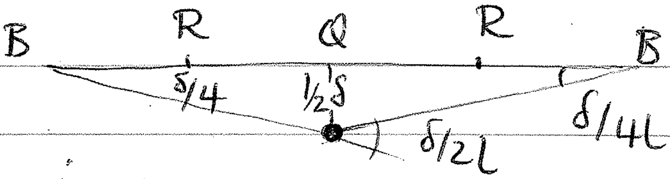
EXAMPLE



Deformation of this grillage must involve twisting of the members, but the corresponding twisting moments will be ignored.

A possible collapse mech. will result if a plastic hinge is developed at the centre of length of each six beams.





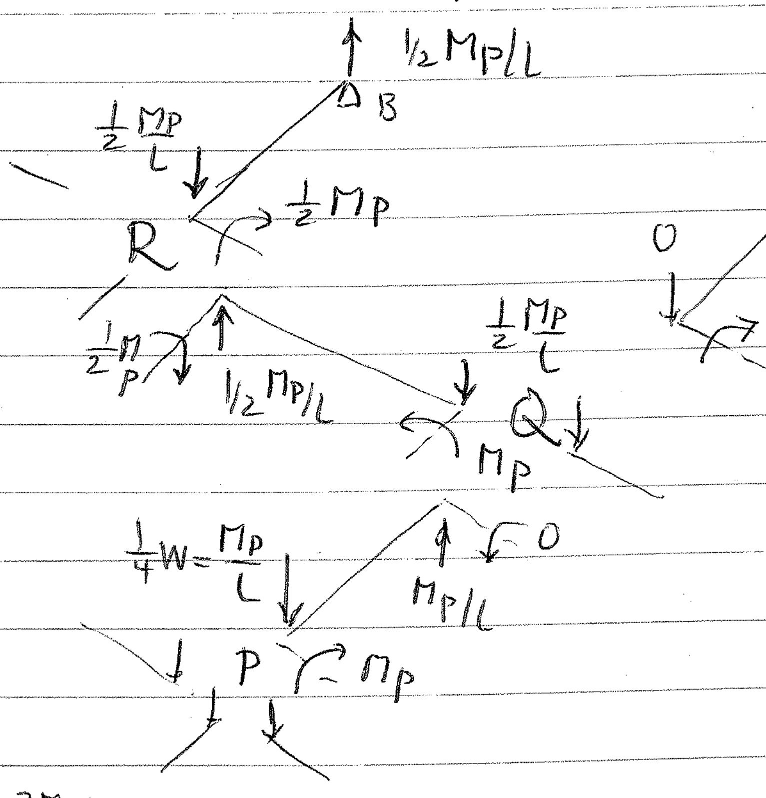
The work equation is as

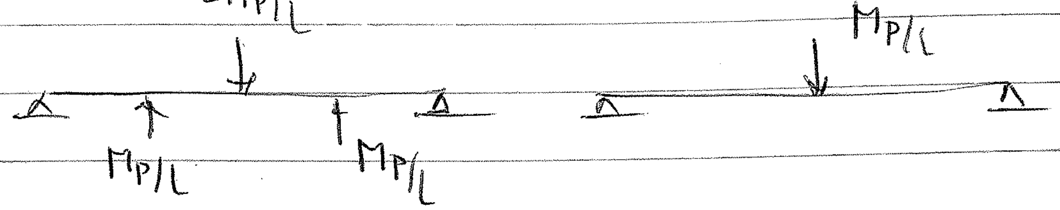
$$W\delta = (2)M_p\left(\frac{\delta}{L}\right) + (4)M_p\left(\frac{\delta}{2L}\right)$$

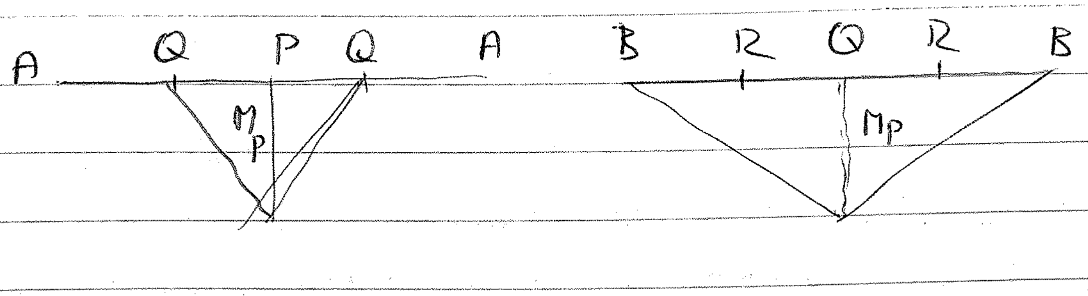
or

$$W = 4M_p/L \qquad (5\text{-}9)$$

~~This value of W must be regarded as an upper bound, pending a statical analysis.~~
The grillage is statically determinate at collapse and forces and moments are shown. No where bending exceeds M_p and therefore (5-9) gives the correct collapse load.



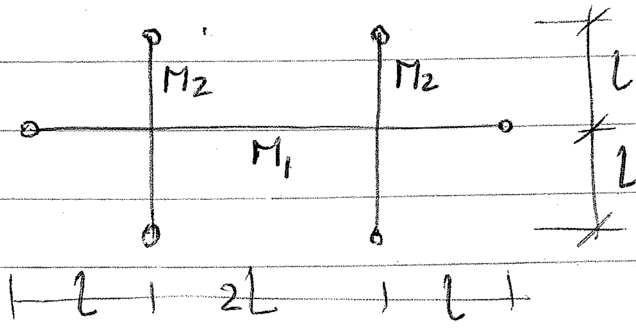




The reader may wish to check that the collapse load of the same grillage, acted upon by a load W at each of the nodes (i.e. total load is $9W$), is given by $W = \frac{M_P}{L}$ and the collapse mech. is unchanged.

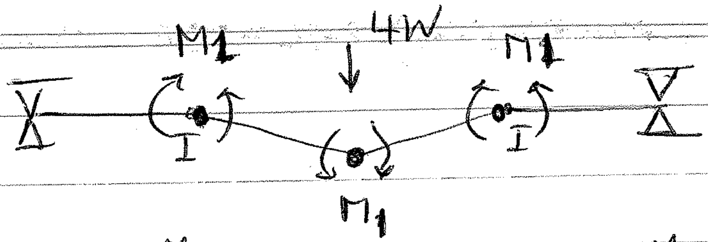
EXAMPLE 2:

For the following grillage, the ends of the beams are pinned to abutments which can resist both upward and downward forces.



The grillage is designed to carry a single load $4W$ placed anywhere on the longer beam. The value of M_2 is such that the short beams do not participate in the collapse mech. M_2 will be found later. The longer beam can be considered as a continuous beam on four supports and it is quickly evident that the most unfavorable position of the point load is at the centre of beam.

(a)



(b)

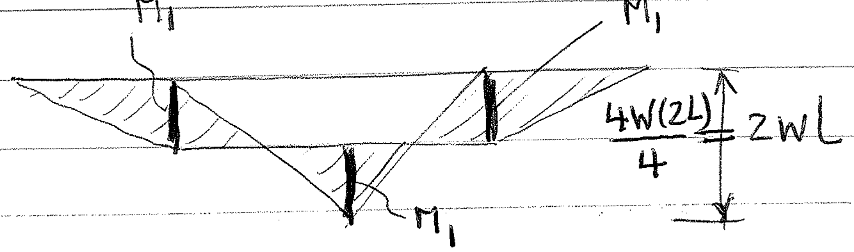


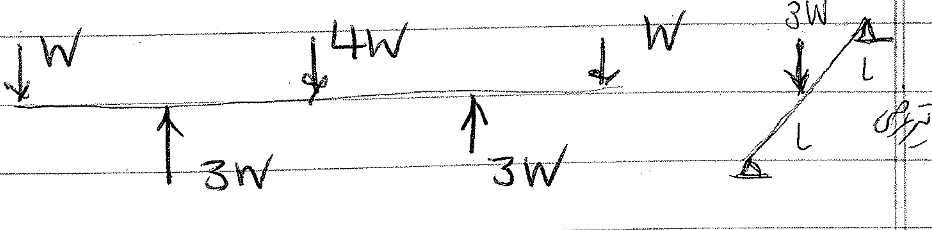
Fig.(5-9)

From B.M diagram M_1 can be found as

$$M_1 = WL \qquad (5\text{-}10)$$

Corresponding to this collapse B.M.D., the abutments must provide downward reactions of value W, so that the load transmitted to each of the shorter beams is 3W.

(c)



Fig(5-9)

The B.M. at the centre of each of shorter beams has value $\frac{3}{2}WL$, so that a possible design would be $M_2 = \frac{PL}{4} = \frac{3W(2L)}{4} = \frac{3}{2}WL$

$$M_2 = \frac{3}{2}WL = \frac{3}{2}M_1 \qquad (5\text{-}11)$$

However, design is not satisfactory when the load 4W moves away from the centre of the grillage.

To see this, the collapse of a grillage with $M_1 = WL$ and $M_2 = \frac{3}{2} WL$ will be investigated.

<u>A possible collapse</u> is shown

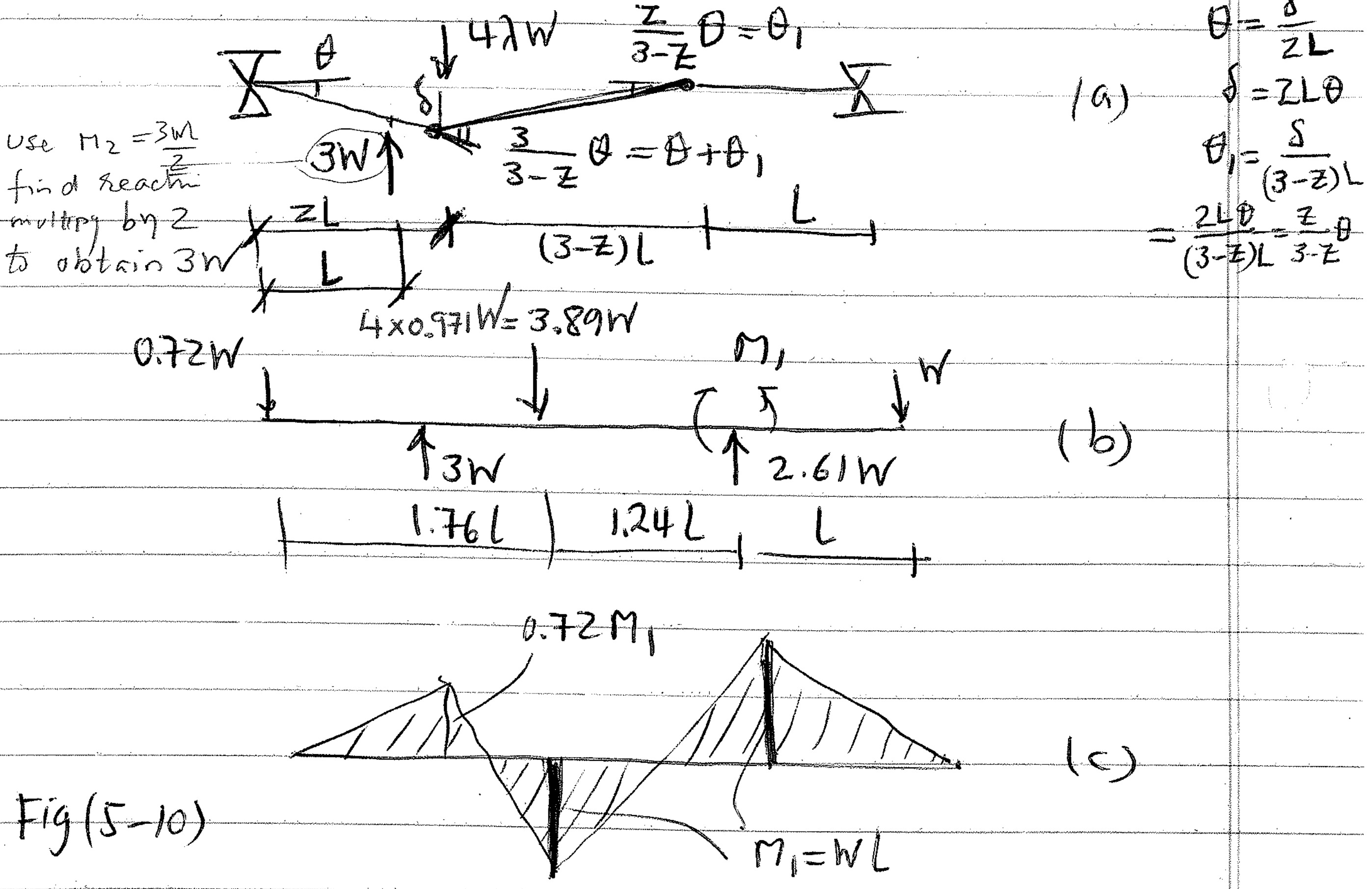


Fig (5–10)

This collapse mech. involves the collapse of one of the shorter beams; the load factor 4W is shown as λ, and the value of the parameter z defining the position of the sagging hinge in the longer beam is still unknown.

Since the shorter beam is collapsing, it must be subjected to 3W for which it was designed, as shown in fig. 5-9(c) and Eq. (5–11).

$$M_1\left(\frac{3}{3-z}\right)\theta + M_1\left(\frac{z}{3-z}\right)\theta \quad 10.$$

The work equation for mech. of Fig. 5-10(a) is

$$4\lambda W(zl\theta) - 3W(l\theta) = M_1\left(\frac{3+z}{3-z}\right)\theta = Wl\left(\frac{3+z}{3-z}\right)\theta$$

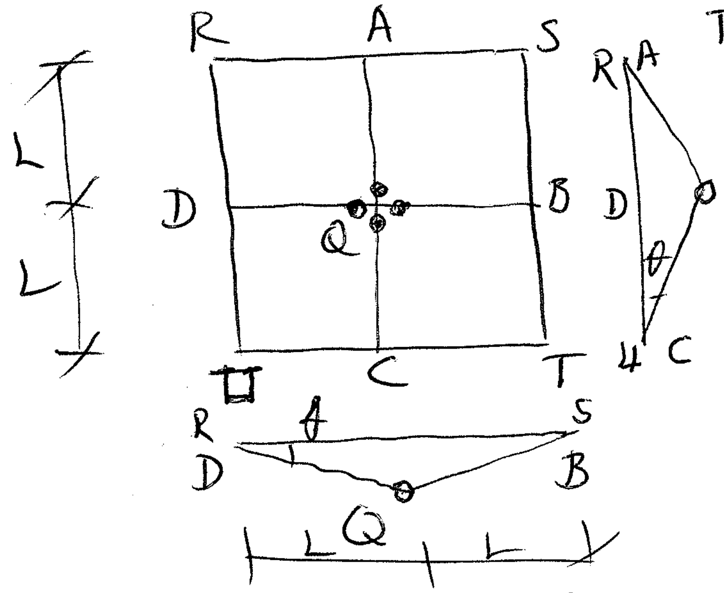
i.e.
$$\lambda = \frac{1}{2}\left[\frac{6-z}{z(3-z)}\right] \qquad (5\text{-}12)$$

λ is minimum for $z = 6-3\sqrt{2} = 1.757$ and the corresponding λ is $\lambda = \frac{1}{6}\left[3+2\sqrt{2}\right] = 0.971$.

The forces for such condition are shown in Fig. (5-10b) and B.M. at Fig. (5-10c).

It will be seen that the B.M. in longer beam satisfy the yield condition and the load on the shorter beam which is not supposed to collapse has value 2.61W which is less than collapse load 3W. Thus the correct solution is found, and the conclusion must be drawn that the grillage designed to have the full plastic moments of (5-10) & (5-11), would not carry the point load 4W traversing the long beam. The load factor under these conditions is 0.971, so M_1 and M_2 were increased by 3%, then the design would be satisfactory. Or M_1 is kept Wl and M_2 is increased 6%.

A 2X2 Grid



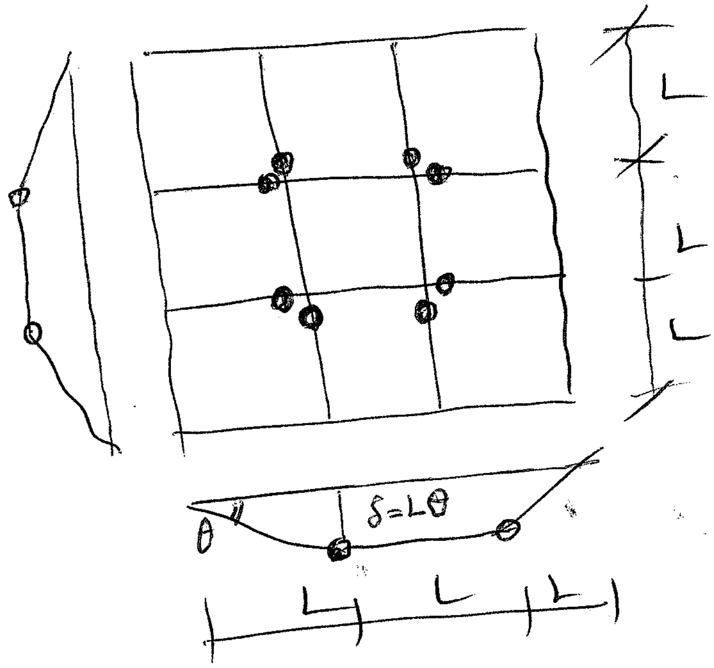
The Two beams AC and DB are simply supported against the frame RSTU. A Concentrated load W is applied perpendicular to the plane of grid at the intersection Q of the Two beams.

From symmetry it follows that no torque will be present. In general, however, beams will transmit both bending and torque. It follows that true equilibrium relations at a joint will be complex. Here we shall ignore any twisting moments by assuming that beams have no resistance to torque. Thus any result we get will be conservative. Heyman has shown that for typical I Sections the error is less than 0.1 per cent for the grids being considered.

The only possible collapse pattern is shown by arrangement of hinges. The deformation profiles of the grid beams are also shown. From work Equation we have $WL\theta = 4M_p\theta$ $\quad W = \frac{4M_p}{L}$

A 3X3 Grid

This is simple. Again there is only one possible symmetric deformation pattern as shown. Here, each of the four load W moves a distance $L\theta$ and 8 hinges rotate an angle θ. Therefore

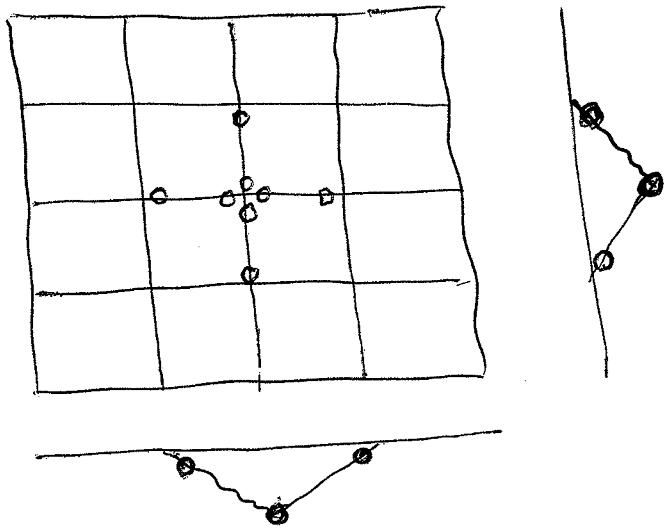


$$4WL\theta = 8M_p\theta$$

$$W = \frac{2M_p}{L}$$

A 4x4 Grid

First mode:

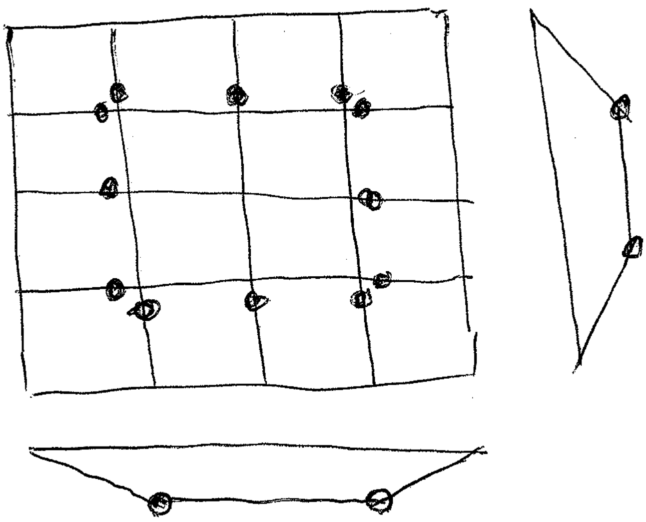


For this case we have

$$WL\theta = 8M_p\theta$$

$$W = \frac{8M_p}{L}$$

Second mode:

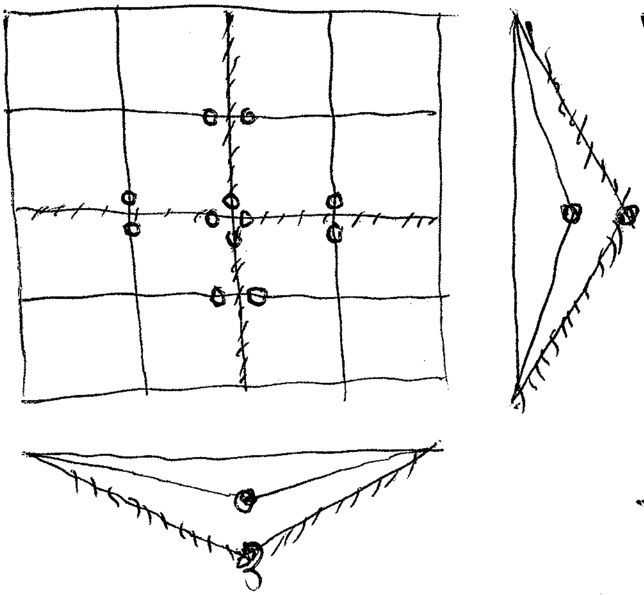


For this case

$$9WL\theta = 12M_p\theta$$

$$W = \frac{4M_p}{3L} = \frac{1.33M_p}{L}$$

Third mode:



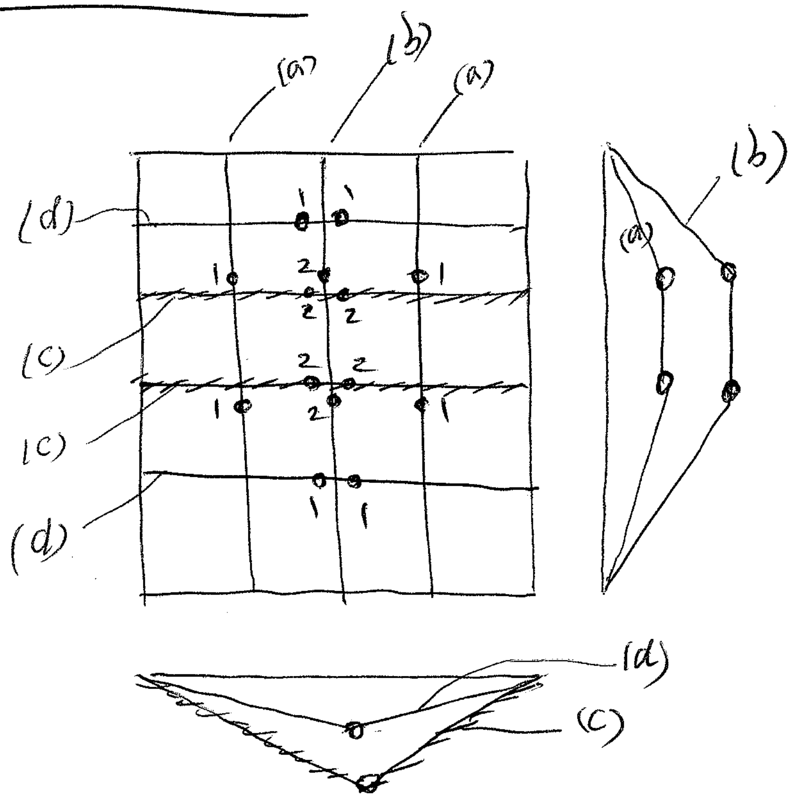
This is a uniform pyramidal deformation. The central node moves $4L\theta$. The corner loads each move $L\theta$ and the remaining four loads move $2L\theta$

$$(8+8)WL\theta = (4+4+8)M_p\theta$$

$$W = \frac{M_p}{L}$$

This is the answer and satisfies equilibrium

A 4x5 Grid



The nodes weighted by "2" rotate 2θ

$$2(2L\theta)(W) + 2(4L\theta)(W) + 2(2L\theta)(W) + (1)(4L\theta)(2W)$$

$$= (6 \times 2 + 8 \times 1) M_p \theta$$

$$24WL\theta = 20 M_p \theta$$

$$W = \frac{20 M_p}{24 L}$$

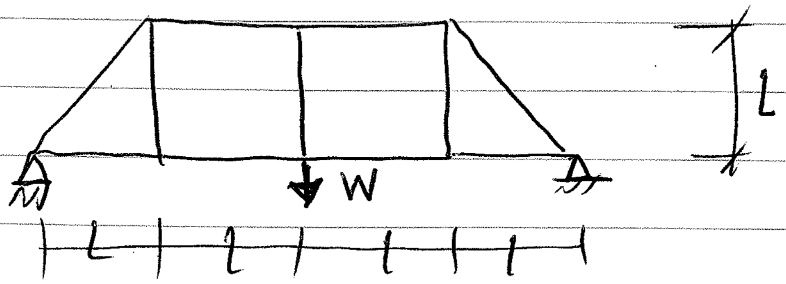
This collapse mode also satifies the equilibrium and is the final answer.

5.3 The Vierendeel Girder

The action of this Girder is to resist loading mainly by bending of members. As usual we assume the axial loads in the member do not cause instability, the effect on M_p can be allowed if necessary.

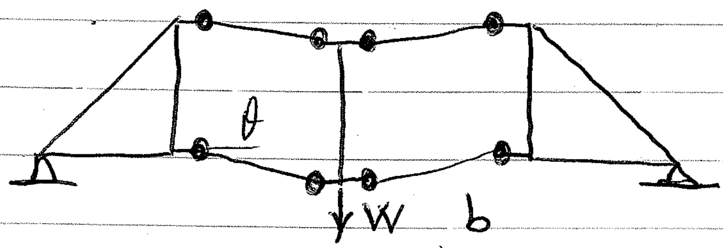
EXAMPLE 1:

M_p for all members



(a)

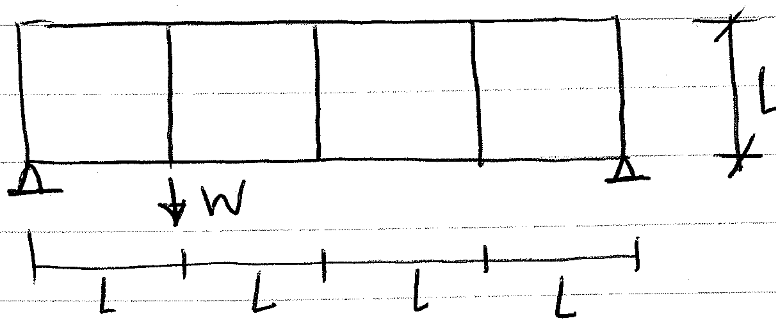
The two ends are translated and can not participate in any collapse mechanism (except possibly as rigid body rotation). A simple mech. is as follows:



for which $Wl\theta = 8M_p\theta$ or $Wl = 8M_p$.

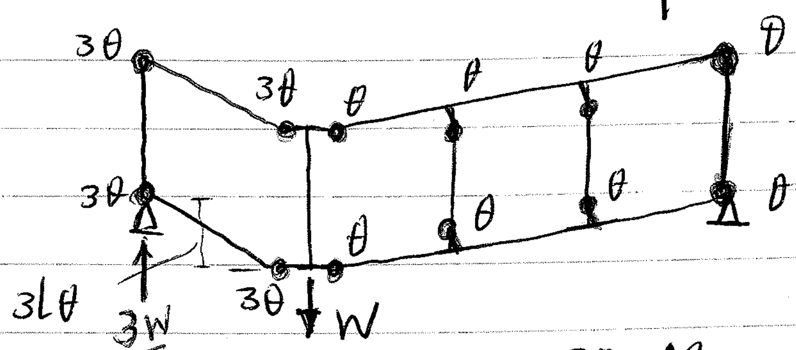
EXAMPLE 2

Mp is assumed for all members



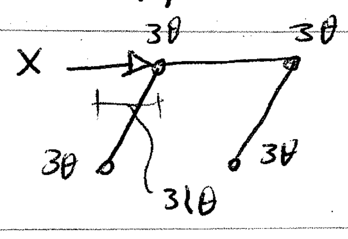
First mech. is considered as (b). This mech. does not correspond to any possible equilibrium state; but it is a possible mode of deformation, which will therefore lead to an upper bound. The collapse equation is

$$3WL = 20M_P \qquad (5\text{-}27)$$



Therefore $$W_c \leq \frac{20}{3}\frac{M_P}{L} \qquad (5\text{-}28)$$

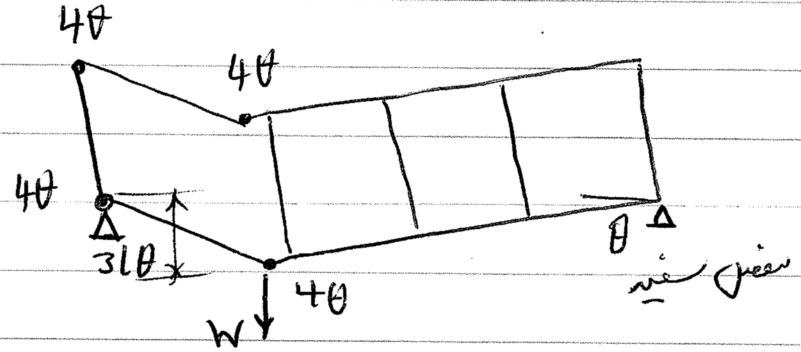
To check the above statement, equil. of the whole girder requires a reaction of $\frac{3}{4}W$ at the left-hand support, that is $\frac{5M_P}{L}$ from Eq. (5-27). However, consider the left-hand bay as a simple square portal frame on its sides, it will be seen that the reaction at the left-hand support should be $\frac{4M_P}{L}$.



$$3L\theta(X) = 4 \times 3\theta M_P$$

$$X = \frac{4M_P}{L}$$

The correct collapse mech. is as follows:

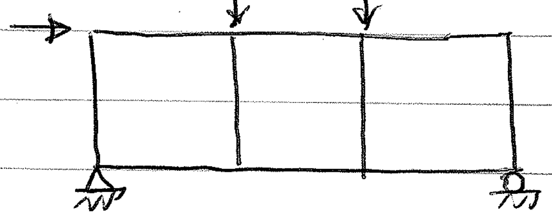


$$3WL = 16M_p$$

$$W_c \le \frac{16}{3}\frac{M_p}{L} \qquad (5\text{-}29)$$

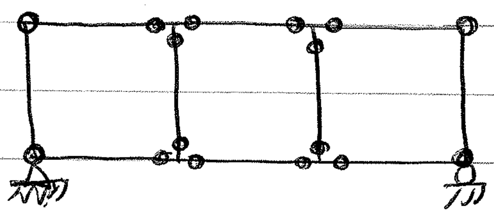
Constructing B.M. shows that yield is not violated and $\frac{16}{3}\frac{M_p}{L} = W_c$.

Example 2



$\gamma(s)=3\times3=9$

$N=16$

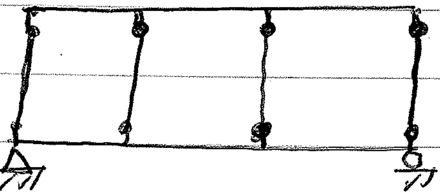


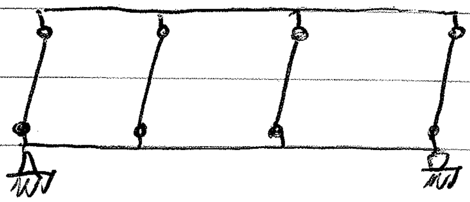
$N=16$

We have $N-\gamma(s)=16-9=7$ basic mechanisms. 4 joint mechanisms and 3 others. Two sets of basic mechanisms are shown in the following:

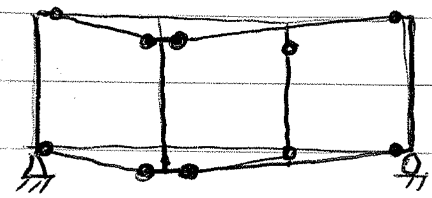
4 joints + 1 sway + 2 beam Or 4 joints + 3 sways (1 sway + 2 panels)

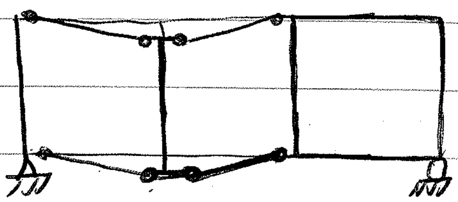
Sway





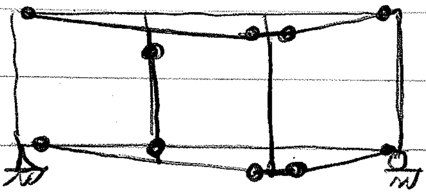
Beam

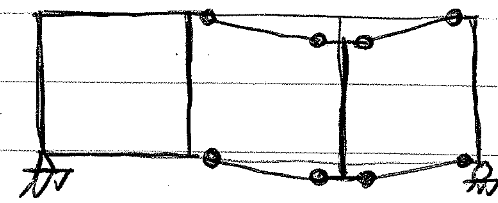




Panel

Beam





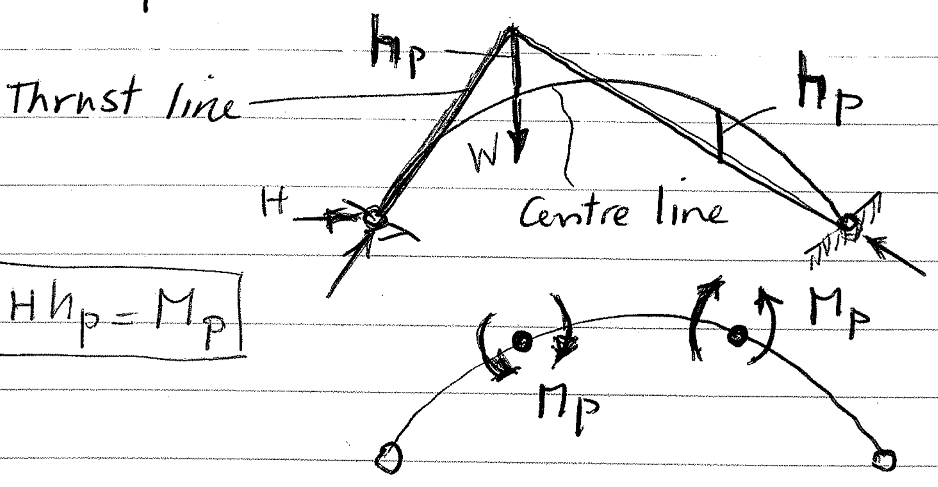
Panel

Basic Mechanisms			Basic Mechanisms		

5.4 The Arch

When the rise of an arch is small the assumptions of simple plastic Theory are violated. First, axial loads in the members of arch Tends to be high: Allowance can be made on Mp, however, instability becomes more critical. Second, and of great importance for shallow arches, the line of Thrust will lie close to The arch rib, i.e. This means That any small deflexion of The arch, ignored in smpl plastic calculations, will have a marked effect on The bending moments; The change of geometry of The arch due to deflexions will upset The equilibrium equations.

For arches which are not too shalow plastic Theory gives good estimate of strength, and a single-span arch is similar to a portal frame. As an example in Fig (5-20) Two plastic hinges are needed to form a collapse mechanism.



$$\boxed{H h_p = M_p}$$

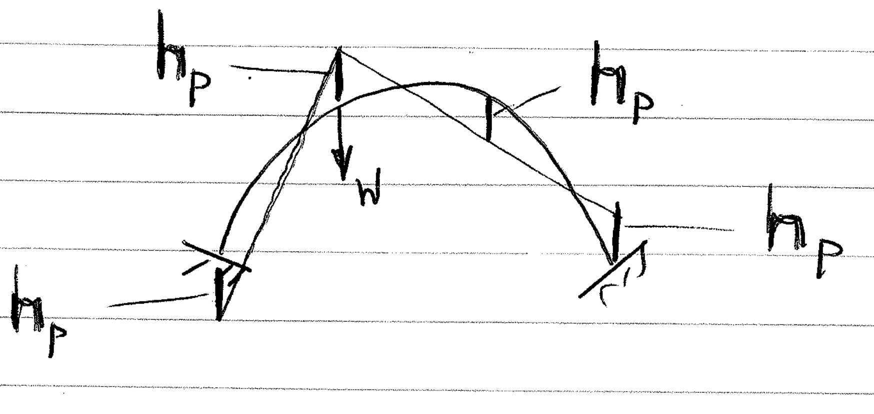
(a) An arch with W

(b) collapse mode

Fig. (5-20)

Proof: Back of previous page.

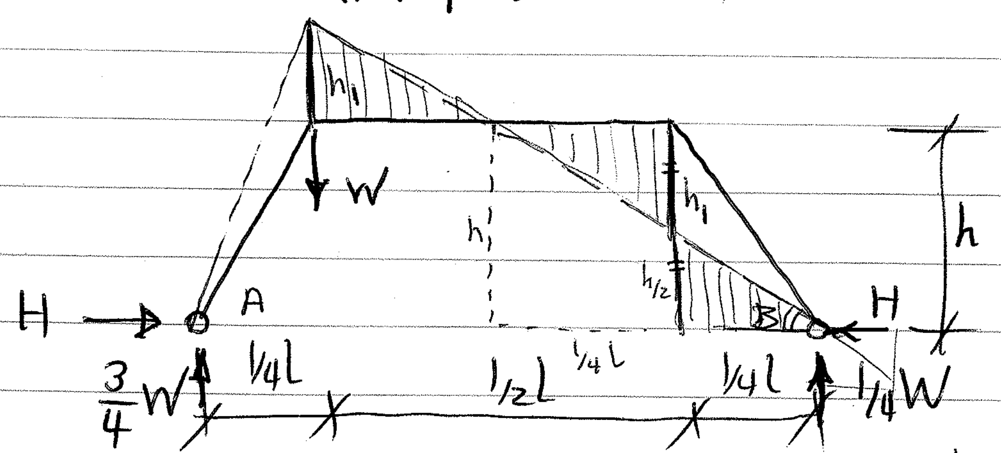
The basic properties hold whether elastic or plastic: The vertical intercept between the thrust line and the centre line of the arch multiplied by the horizontal component of the abutment thrust, gives the value of the bending moment at any section. Thus the two intercepts in Fig. 5-20a must be made equal (for plastic case M_p). Similarly, the same arch with fixed abutments will have thrust line at collapse as shown in Fig. 5.21



$h_p \times H = M_p$

H: horizontal component of the abutment thrust.

NUMERICAL EXAMPLE



$$\frac{H}{\frac{W}{4}} = \frac{\frac{L}{4}}{h/2} \Rightarrow Hh = \frac{WL}{8}$$

Thrust line should be positioned to give equal intercepts h_1. By simple geometry

$$h_1 = \frac{1}{2}h$$

The value of H can be found by triangle of equil. at B.

$$\frac{H}{\frac{1}{4}W} = \frac{\frac{1}{4}l}{\frac{1}{2}h}$$

or

$$Hh = \frac{1}{8}Wl \qquad (5\text{-}30)$$

Thus if M_p is the full plastic moment by definition and using (5-30),

$$M_p = Hh_1 = \frac{1}{2}Hh = \frac{1}{16}Wl \qquad (5\text{-}31)$$

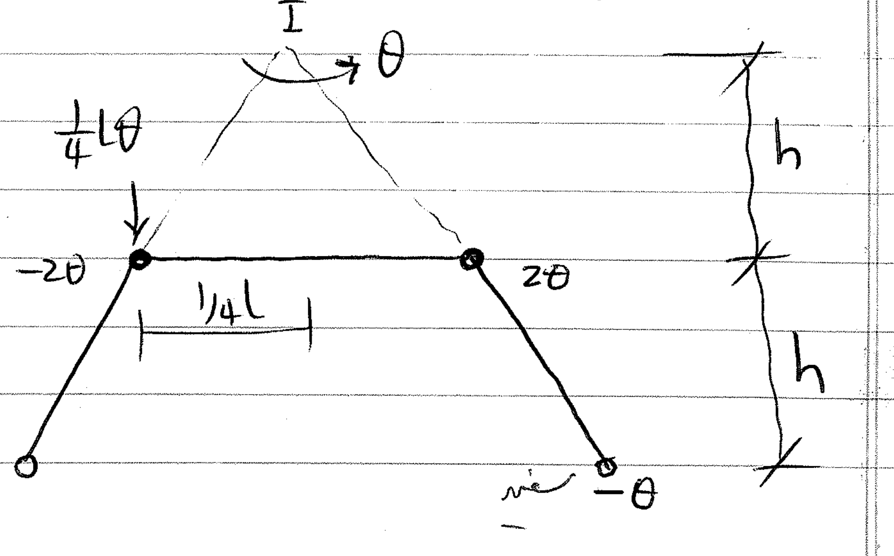
We could find the same result from work equation

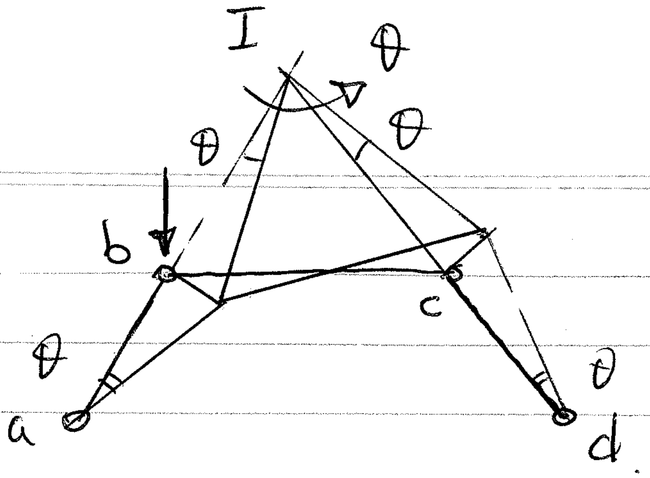
$$W\left(\frac{1}{4}l\theta\right) = M_p(4\theta) \qquad (5\text{-}32)$$

Similarly for mech. of Fig.(5-23) for the fixed base arch, leads to collapse eq. as

$$\frac{1}{4}Wl = 4M_p$$

(5-33)





Rotation of b = θ (for ab) + θ (for bc) = 2θ
Rotation of c = θ (for bc) + θ (for cd) = 2θ

for b we have − 2θ opened
for c we have + 2θ closed.

The use of a thrust line in the analysis of arches is analogous to the use of free and reactant B.M. diagrams for beams & frames.

Once again, therefore, there is choice between working from the principle of statics in constructing a thrust line, or using the work equation associated with a mech. of collapse. Very often a mixture of the two approaches will give the deepest insight into structural behaviour, and will lead to the quickest calculations.