3D BENCHMARK RESULTS FOR ROBUST STRUCTURAL OPTIMIZATION UNDER UNCERTAINTY IN LOADING DIRECTIONS

A. Csébfalvi[*, †]

Department of Structural Engineering, University of Pécs, Boszorkány 2, Pécs, Hungary

ABSTRACT

This study has been inspired by the paper "An efficient 3D topology optimization code written in MATLAB" written by Liu and Tovar (2014) demonstrating that SIMP-based three-dimensional (3D) topology optimization of continuum structures can be implemented in 169 lines of MATLAB code. Based on the above paper, we show here that, by simple and easy-to-understand modifications we get a few lines longer code, which is able to solve robust topology optimization problems with uncertain load directions. In the presented worst load direction oriented approach, the varying load directions are handled by quadratic constrains, which describe spherical regions about the nominal loads. The result of the optimization is a robust compliance-minimal volume constrained design, which is invariant to the investigated directional uncertainty. The key element of the robustification is a worst-load-direction searching process, which is formulated as a small quadratic programming problem with quadratic constraints. The presented approach is a 3D extension of the robust approach originally developed by Csébfalvi (2014) for 2D continuum structures. In order to demonstrate the viability and efficiency of the extension, we present the model and algorithm with detailed benchmark results for robust topology optimization of 3D continuum structures. It will be demonstrated that the computational cost of the robustification is comparable with its deterministic equivalent because its central element is a standard 3D deterministic multi-load structure optimization problem and the worst-load-direction searching process is formulated as a significantly smaller quadratically constrained quadratic programming problem, which can be solved efficiently by several different ways.

---

[*]Corresponding author: Department of Structural Engineering, University of Pécs, Boszorkány 2, Pécs, Hungary
[†]E-mail address: csebfalvi.witch@gmail.com (A. Csébfalvi)

1. INTRODUCTION

In the real-world topology optimization problems, the optimal performance obtained using conventional deterministic methods can be dramatically degraded in the presence of sources of uncertainty. The source of uncertainty may be the variability of applied loads, spatial positions of nodes, material properties, and so on. Various probabilistic approaches have been developed to account for different types of uncertainty in structural design and optimization methods (see, for example, Choi et al. [3], however, the paradigm had not been applied to structural topology optimization methods until recently. The interested reader is directed to Bendsøe and Sigmund [4], which contains an extensive bibliography on this subject.

At present there are two main approaches that consider uncertainties in structural topology optimization. The first is to introduce a number of reliability constraints based on probability of failure, often referred to as reliability-based topology optimization (RBTO) (see, for example, Kharmanda [5]. These methods often aim to minimize the weight or volume of a structure, while ensuring the probability of failure is less than a prescribed value. The failure probability constraint is a function of the uncertain parameters and is usually recast to ensure the reliability of the structure is greater than a required value. When a failure state is not defined, an alternative approach to RBTO is to consider a probabilistic objective that is a function of the uncertain parameters. This approach is often referred to as robust optimization. Popular choices for robust objective functions are to minimize the expected or maximum performance and both approaches have been used when solving the classic minimization of compliance problem with uncertain variables (see Califoire et al. [6] and de Gournay et al. [7]). Various parameters can affect the robustness and reliability of a structure, including loading, geometry and material properties. Loading uncertainties are most widely studied, although the level of uncertainty is often limited to loading magnitude. However, uncertainty in loading direction can be simulated by considering orthogonal uncertain loads with zero mean (Mogami et al. [8]). This approach may be appropriate for some problems, however, the orthogonal loads are often uncorrelated and the relevance to the directional uncertainty of a single load can be unclear. For non-probabilistic uncertainties, loading direction has been considered using the multi-ellipsoid convex model (Kang and Luo [9]), which does allow for some interaction between orthogonal loads. For the robust optimization approach, methods for approximating probabilistic directional uncertainties include using a discrete probability function (Conti et al. [10]), discretization of a continuous probability function using a sampling method (Calafiore et al. [11] and Evgrafov et al. [12]) and an approximation by a quadrature technique (Chen et al. [13]). These discretized approaches transform the optimization problem into a multiple load case one, which can be solved for the minimization of expected performance problem. Recently Dunning et al. [14] proposed an efficient method for considering loading magnitude and directional uncertainty in topology optimization in order to produce robust solutions. The classic minimization of compliance problem is considered with uncertainties being introduced into the objective function and described by continuous normal probability functions. In another paper, Dunning et al. [15] presented a new model for simultaneous minimization of expectancy and variance of compliance in the presence of uncertainties in loading magnitude using exact formulations and analytically derived sensitivities. Recently, Liu and Tovar [16] demonstrated that SIMP-based three-dimensional (3D) topology optimization of continuum structures can be

easily implemented in 169 lines of MATLAB code, which is an extension of the 88 line long MATLAB code developed by Andreassen et al. [17] for 2D topology optimization.

A new worst-load-direction oriented approach for robust topology optimization with uncertain-but-bounded load directions was introduced by Csébfalvi [18]. 2D benchmark results for the proposed approach were presented by Csébfalvi [19]. This study has been inspired by the paper "An efficient 3D topology optimization code written in MATLAB" written by Liu and Tovar [16] demonstrating that SIMP-based three-dimensional (3D) topology optimization of continuum structures can be easily implemented in 169 lines of MATLAB code. Based on the above paper, we show here that, by simple and easy-to-understand modifications we get a few lines longer MATLAB code from the original 169 line MATLAB code, which is able to solve 3D robust topology optimization problems with uncertain load directions.

The paper is organized as follows. The model of robust topology optimization of 3D continuum structures is presented in Section 2, where we show that the result is a compliance-minimal design for a given volume constraint, which is invariant to the directional uncertainty. The central element of the robust solution searching process is a standard deterministic multi-load optimization model developed for solid isotropic material. The worst-load-direction searching model is formulated as a small quadratic programming problem with quadratic and box constraints. In Section 3, we describe the algorithm of the robust 3D topology optimization model. In Section 4, detailed and well-illustrated benchmark results will be shown to demonstrate the vitality and efficiency of the robust topology optimization of 3D continuum structures. Finally, some concluding remarks are presented in Section 5.

2. MODEL

The mathematical formulation of the deterministic 3D topology optimization of continuum structures can be formulated as follows:

$$c(\boldsymbol{x}) = \boldsymbol{U'KU} \rightarrow \boldsymbol{min} \tag{1}$$

$$V(\boldsymbol{x}) = \mu V_0 \tag{2}$$

$$\boldsymbol{KU} = \boldsymbol{F} \tag{3}$$

$$\boldsymbol{0} \le \boldsymbol{x} \le \boldsymbol{1} \tag{4}$$

where c is the compliance, $\boldsymbol{U}$ and $\boldsymbol{F}$ are the global displacement and force vectors, respectively, $\boldsymbol{K}$ is the global stiffness matrix, $\boldsymbol{x}$ is the vector of design variables (the element densities), $V(\boldsymbol{x})$ and V_0 are the material volume and design domain volume, respectively, and μ is the prescribed volume fraction. The design domain is assumed to rectangular and discretized with eight-noded cubic elements with three degrees of freedoms (DOFs) per node. The nodes identified with a number ordered column-wise up-to-bottom, left-to-right, and back-to-front. The position of each node is defined with respect to Cartesian coordinate system with origin at the left-bottom-back corner. The optimization problem (1) can be solved by, for example, the well-known optimality criteria method, but there are several other possibilities to manage the problem (see, for example, Liu and Tovar [16]).

It is very easy to extend the algorithm to account for multiple load cases (see, for example, Andreassen et al. [17]). In fact, this can be done by adding only a few additional lines and making minor changes to another few lines. In the case of m load cases, the force and displacement vectors can be defined as m column vectors and the objective function will be the sum of m compliances:

$$c(x) = \sum_{i=1}^{m} U_i' K U_i \rightarrow min \tag{5}$$

$$V(x) = \mu V_0 \tag{6}$$

$$KU_i = F_i, \; i \in \{1, 2, \ldots, m\} \tag{7}$$

$$0 \le x \le 1 \tag{8}$$

Now, we will show that the multi-load compliance-minimization model, after simple modifications can be used to solve our directional uncertainty problem (see Csébfalvi [3]). Let $F = F(\varphi)$, where $0 \le \varphi \le \overline{\varphi}$, denotes a load vector with uncertain load directions, which form a spherical region around the nominal load direction. In Fig. 1, the nominal load direction is represented by a solid arrow, which may be perturbed by maximum $|\varphi| \le 30°$ in any direction.
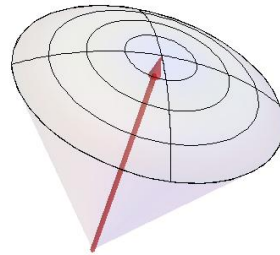

Figure 1. A spherical region around the nominal load

First, we rewrite the deterministic single load optimization model according to the uncertain load directions. The theoretical formulation of the modified optimization problem can be described as follows:

$$c(x) = U'KU \rightarrow min \tag{9}$$

$$V(x) = \mu V_0 \tag{10}$$

$$KU = F(\varphi) \tag{11}$$

$$0 \le \varphi \le \overline{\varphi} \tag{12}$$

$$0 \le x \le 1 \tag{13}$$

According to the modification, the goal is to minimize the compliance of the structure for all the feasible loads $F(\varphi)$, where $0 \le \varphi \le \overline{\varphi}$. It is worth noting that after inserting the $0 \le \varphi \le \overline{\varphi}$ relation we get a mathematical program with infinite number of constraints since set φ is in general a set of continuous sets with uncountable infinite number of elements.

The result of the optimization will be a robust compliance-minimal design for the prescribed μ volume fraction, which is invariant to the investigated load uncertainty type. As we are only interested in linear elastic structures, the stiffness matrix K and its inverse K^{-1} are symmetrical.

Let us denote by $p = |\varphi|$ the number of the external point loads with directional uncertainty. Here we assume that all uncertain variables φ_i, $i \in \{1, 2, \ldots, p\}$ are statistically independent. A directionally uncertain 3D point load with magnitude f_i, $i \in \{1, 2, \ldots, p\}$ can be written in terms of three orthogonal loads. For example, when we assume that all the loads with directional uncertainty are defined in the z direction, than they can be described by the following three spherical coordinates:

$$f_i = \begin{cases} f_i^x(f_i, \varphi_i) = f_i \, cos(\theta_i) sin(\varphi_i) \\ f_i^y(f_i, \varphi_i) = f_i \, sin(\theta_i) sin(\varphi_i) \\ f_i^z(f_i, \varphi_i) = f_i \, sin(\theta_i) cos(\varphi_i) \end{cases}, 0 \le \theta_i \le 2\pi, i \in \{1, 2, \ldots, p\} \tag{14}$$

Naturally, the spherical coordinates can be replaced by their Cartesian equivalents very easily to simplify the description of the spherical regions in the worst load direction searching quadratically constrained $3p \times 3p$ quadratic programming problem.

3. ALGORITHM

In order to avoid dealing with infinite number of constraints we have to replace the original problem with a more tractable equivalent algorithm based on a finite number of constraints. The essence of our "minimal-art" algorithm is very simple:

1. First, we redefine the nominal problem, as it would be a multiple load problem with currently one load case, therefore we start with $m = 1$ and $F_1 = F$, where F is the deterministic nominal load vector, and set the iteration counter i to zero.
2. We solve the current multiple load compliance minimization problem subject to the multiple load constraint sets, where the objective function is defined as the sum of the compliances and its sensitivity is defined as the weighted sum of the sensitivities with the compliances as weights.
3. Using the stiffness matrix of the previously given best solution we maximize the compliance on the set of the feasible load directions. The result will be the worst load vector on the set

$0 \leq \varphi \leq \overline{\varphi}$. If the iteration counter i is zero then we replace the nominal load vector by the worst load vector and its possible counterparts according to the identified structural symmetricity, increase the iteration counter by one, and repeat Step 2. Otherwise, we check if the stopping criterion is satisfied. If it is true then the algorithm terminates with the best robust solution. Otherwise, we increase the iteration counter by one and test the membership of the worst load vector in the multiple-load set. If the worst load vector is not a set member then we increase m by one and append the vector to the multiple-load set. If the worst load vector has possible counterparts according to the structural symmetricity we repeat the increasing and adding process for all counterparts. After that, independently from test result, we repeat Step 2.

In the implementation of the algorithm we have several different ways to manage Step 3 from an appropriate sampling technique based on a mesh and interpolation to the global solution of a large nonlinear optimization problem. We have to note, that, in this context "large" means that number of uncertain-but-bounded parameters is large. When the danger of "combinatorial explosion" is negligible, then the direct evaluation of the compliance on a mesh which is fine enough may be the most safety and most efficient "global" problem handling strategy. Theoretically, the worst compliance searching process can be described in the following form:

$$c(\mathrm{x}) = \mathrm{F}(\varphi)' \mathrm{K}^{-1} \mathrm{F}(\varphi) \to \max \qquad (15)$$
$$0 \leq \varphi \leq \overline{\varphi} \qquad (16)$$

In this study, we measured the changing of the robustness from iteration to iteration with the relative percentage deviation measure ($\textbf{rpd}$), which can be defined in the following forms:

$$\textbf{rpd}(\textbf{\textit{c}}_i, \textbf{\textit{c}}_{i-1}) = 100 \times \frac{\overline{c}_i - \overline{c}_{i-1}}{\overline{c}_{i-1}} \% \qquad (17)$$

where $c_i\left(c_{i-1}\right)$ and $\overline{c}_i\left(\overline{c}_{i-1}\right)$ mean the compliance region and the maximal compliance of the region in iteration $i(i-1)$, respectively. Naturally, it is a very interesting open question, that what would be the best measure which would be able to characterize appropriately the robustness and its change from iteration to iteration, when the volume fraction μ is fixed.

4. EXAMPLE

The example illustrating the 3D topology optimization of continuum structures shown in Fig. 2, is a cantilever beam, where an external unit point load acting in the end-middle-top position of the beam upwards, which is denoted by f and its nominal load direction is $\varphi = 0^\circ$. We suppose, that the spherical region around the nominal load, which defines the feasible load directional perturbations, is defined by the following relation: $0 \leq \varphi \leq 30^\circ$. The Young's

modulus is $E_0 = 1$, the Poisson's ratio is $v = 0$, and the fixed volume fraction is $\varphi = 0.25$. The penalization power is $p = 3$ and we applied sensitivity filtering with filter radius $r_{min} = 1.5$. In this paper, during the robust topology optimization process, a ground structure of $12\,cm \times 8\,cm \times 4\,cm$ was used with $\boldsymbol{rpd}\left(\boldsymbol{c}_i, \boldsymbol{c}_{i-1}\right) \leq 1\%$ setting.
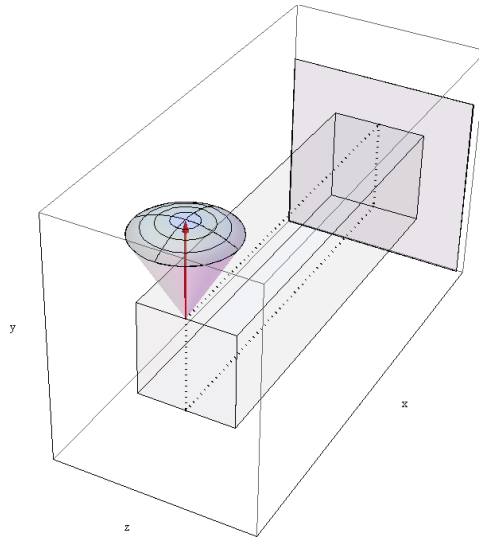
Figure 2. The design domain, boundary conditions, and a single point load with directional uncertainty

In this example, our goal is the following: we try to find a compliance-minimal robust solution for the fixed volume fraction $\mu = 0.2$, which is invariant to the load directional uncertainty. The process terminates after the third iteration with $\boldsymbol{rpd}\left(\boldsymbol{c}_3, \boldsymbol{c}_2\right) = 0.26\%$. The nominal and optimal robust structure shapes are shown in Figs 3-4.
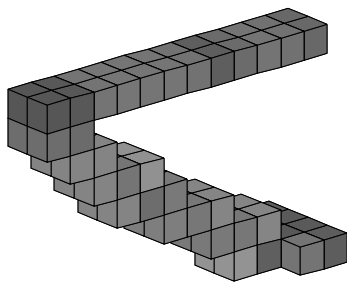

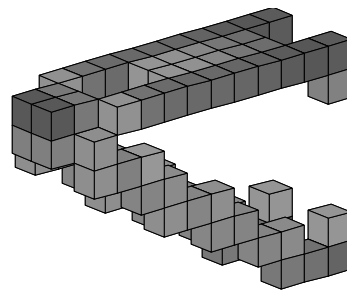Figure 3. The optimal nominal structure shape   Figure 4. The optimal robust structure shape

The plots of the nominal and robust compliances and their common plot on the set of

feasible directions (see Figs. 5-7) well demonstrate the effect of the robust optimization. The robust optimization process changes the nominal shape and the nominal compliance region drastically. It is interesting fact, that the investigated problem is symmetric for the plane, which is visualized by the dotted lines in Fig. 2, therefore each robust iteration adds two worst-loads to the multi-load problem. The multi-load problem which we have to solve in the first iteration is shown in Fig. 8.
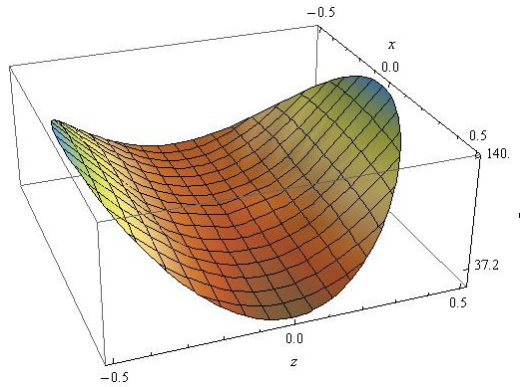

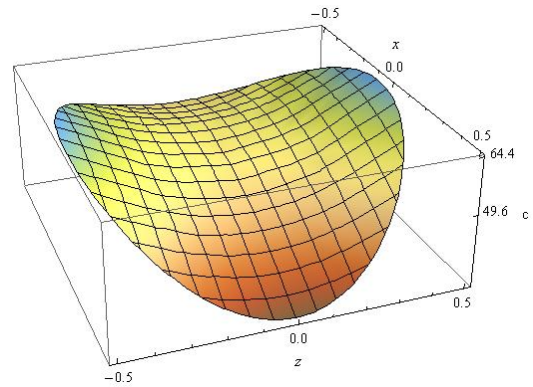
Figure 5. The nominal compliance region

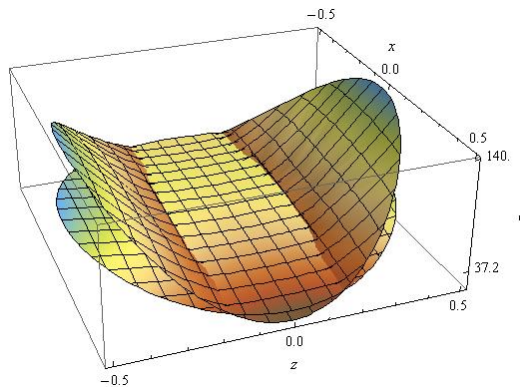Figure 6. The optimal robust compliance region

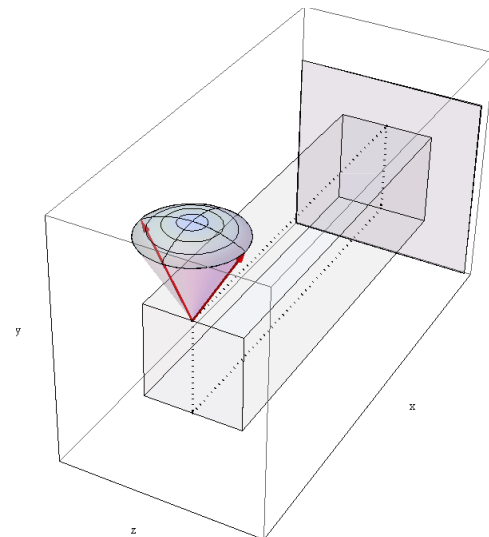Figure 7. The common plot of the nominal and robust compliance regions

Figure 8. The multi-load problem of the first iteration

We could be note, that the results good demonstrate the fact, that the robust design searching process, using the presented simple and easy-to-understand rearrangement strategy, finds a really robust solution very quickly for a given volume fraction μ. It is a very interesting feature of our worst-case-minimizing approach that, according to the constant volume fraction $\mu = 0.2$, the decreasing worst compliance also decreases the compliance range as a useful side effect.

5. CONCLUSION

This study has been inspired by the paper "An efficient 3D topology optimization code written in MATLAB" written by Liu and Tovar [1] demonstrating that SIMP-based three-dimensional (3D) topology optimization of continuum structures can be easily implemented in 169 lines of MATLAB code. Based on the above paper, we shown here that, by simple and easy-to-understand modifications we get a few lines longer MATLAB code from the original 169 line MATLAB code, which is able to solve robust topology optimization problems with uncertain load directions. In the presented worst load direction approach, the varying load directions are handled by quadratic constrains, which describe spherical regions around the nominal loads. Naturally the applied load direction handling method can be replaced by any other uncertainty representation form, which can be described by an appropriate combination of linear or quadratic relations. The result of the optimization is a robust compliance-minimal design for a given volume constraint which is invariant to the investigated loading directional uncertainty. The key element of the proposed robust optimization algorithm is a worst-load-direction searching model, which can be formulated as a small quadratic programming problem with quadratic constraints and box constraints. The presented three-dimensional approach is an extended version of a worst-load-direction-oriented robust optimization approach originally developed by Csébfalvi [18,19] for 2D continuum structures. In order to demonstrate the viability, variability and efficiency of the extended robust approach, we presented the extended model and algorithm with detailed and well-illustrated benchmark results for robust topology optimization of 3D continuum structures. It was be demonstrated that the computational cost of the robustification is comparable with its deterministic equivalent because the central element of the proposed approach is the standard 3D deterministic multi-load structure optimization problem and the worst-load-direction searching algorithm can be formulated as a significantly smaller quadratically constrained quadratic programming problem, which can be solved efficiently by several different ways. The conception is independent from the applied modelling frame, therefore, the presented compliance-oriented "academic" approach could be replaced by a weight-oriented "engineering" approach with density and stress constraints, which will be shown in a forthcoming paper.

REFERENCES

1. Liu K, Tovar A. An efficient 3D topology optimization code written in Matlab, *Struct Multidiscip Optim* 2014; **50**: 1175–96.
2. Sigmund O. A 99 line topology optimization code written in MATLAB, *Struct Multidiscip Optim* 2001; **21**(2): 120-27.
3. Andreassen E, Clausen A, Schevenels M, Lazarov BS, Sigmund O. Efficient topology optimization in MATLAB using 88 lines of code, *Struct Multidiscip Optim* 2011; **43**(1): 1-16.
4. Choi SK, Grandhi RV, Canfield RA. *Reliability-Based Structural Design*, Springer, London, 2007.

5. Bendsøe MP, Sigmund O. *Topology Optimization: Theory, Methods and Applications*, Springer–Verlag, Germany, 2004.

6. Kharmanda G, Olhoff N, Mohamed A, Lemaire M. Reliability-based topology optimization, *Struct Multidiscip Optim* 2004; **26**(5): 295-307.

7. Calafiore GC, Dabbene F. Optimization under uncertainty with applications to design of truss structures, *Struct Multidiscip Optim* 2008; **35**(3): 189-200.

8. de Gournay F, Allaire G, Jouve F. Shape and topology optimization of the robust compliance via the level set method, ESAIM: *COCV* 2008; **14**(1): 43-70.

9. Mogami K, Nishiwaki S, Izui K, Yoshimura M, Kogiso N. Reliability-based structural optimization of frame structures for multiple failure criteria using topology optimization techniques, *Struct Multidiscip Optim* 2006; **32**(4): 299-311.

10. Kang Z, Luo Y. Non-probabilistic reliability-based topology optimization of geometrically nonlinear structures using convex models, *Comput Meth Appl Mecha Eng*; 2009; **198** (41-44): 3228-38.

11. Conti S, Held H, Pach M, Rumpf M, Schultz R. shape optimization under uncertainty: a stochastic programming perspective, *SIAM J Optim* 2009; **19**(4): 1610-32.

12. Evgrafov A, Patriksson M. Stochastic structural topology optimization: discretization and penalty function approach, *Struct Multidiscip Optim* 2003; **25**(3): 174-88.

13. Chen S, Chen W, Lee S. Level set based robust shape and topology optimization under random field uncertainties, *Struct Multidiscip Optim* 2010; **41**(4): 507-24,

14. Dunning P, Kim HA, Mullinuex G. Introducing Loading Uncertainty in Topology Optimization, *AIAA J* 2011; **49**(4): 760-68.

15. Dunning PD, Kim HA. Robust topology optimization: Minimization of expected and variance of compliance, *AIAA J* 2013; **51**(11): 2656-64.

16. Liu K, Tovar A. An efficient 3D topology optimization code written in Matlab, *Struct Multidiscip Optim* 2014; **50**: 1175-96.

17. Andreassen E, Clausen A, Schevenels M, Lazarov BS, Sigmund O. Efficient topology optimization in MATLAB using 88 lines of code, *Struct Multidiscip Optim* 2011; **43**(1): 1-16.

18. Csébfalvi A. Robust topology optimization with uncertain-but-bounded load directions. *Struct Multidiscip Optim* (submitted), 2014.

19. Csébfalvi A. Robust topology optimization algorithm: benchmark results. in *The Twelfth International Conference on Computational Structures Technology*, B.H.V. Topping and P. Iványi, (Editors), Civil-Comp Press, Stirlingshire, United Kingdom, paper 175, doi:10.4203/ccp.106.175, 2014.