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FORM-FINDING USING OPTIMIZATION-BASED FORCE DENSITY METHOD

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ABSTRACT

Force Density Method is a well-known form-finding method for discrete networks that is based on geometrical equilibrium of forces and could be used to design efficient structural forms. The choice of force density distribution along the structure is mostly upon user which in most cases is set be constant, with peripheral members having relatively larger force density to prevent excessive shrinking. In order to direct FDM towards more efficient structures, an optimization strategy can be used to inform the form-finding process by minimizing certain objective function, e.g. weight of the structure. Desired structural, constructional or geometrical constraints can also be incorporated in this framework that otherwise user may not have direct control over. It has been shown that considerable weight reduction is possible compared to uniform force density in the structure while satisfying additional constraints. In this way, form-finding can be augmented and novel structural forms can be designed.

Keywords: form-finding; force density method; structural optimization; grid shell.

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1. INTRODUCTION

Gridshells are among efficient structural systems that make best use of material to span over large areas. They are in a class of structures in which there is a close relation between form and force, like shells, membranes, cable networks, and tensegrities. These structures cannot be dealt with like conventional ones, and their forms have to be determined with regard to the flow of the forces within them. Form-finding is a stage which is performed

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independently before analysis to determine overall geometry under certain applied loads. Here, the geometry is unknown in advance and a form-finding process is required [1].

Structural optimization has been a topic of much research, mostly in relation to size, shape and topology optimization of the structures [2, 3], and primarily for skeletal structures [4, 5]. Various optimization methods have been developed so far, with two main class of gradient-based and evolutionary-based algorithms. The former requires gradient or Hessian matrix of the objective function that render it unsuitable for complex problem that have non-convex search space and/or have singularities. The latter uses a stochastic approach in searching design space and has proved to perform well in various structural engineering applications [6]. Although structural optimization can be used as a form-finding tool [7], their wide application has happened only after significant improvement in computational capacity of computers occurred. Until 60s, form-finding methods were limited to physical-based modelling, with famous examples like works of Antoni Gaudi, Heinz Isler and Feri Otto. Today, increasing computational capacity has enabled us to perform numerical simulations for form-finding, among which finite element (FE) based methods, Dynamic Relaxation (DR) and Force Density Method (FDM) are more recognized and used. A through review of form-finding methods can be found in [8].

FDM is a well-known form-finding method initially proposed for pin-jointed networks in tension. Introducing a quantity called force density (q) which is equal to the force of the element divided by its length, a nonlinear set of equilibrium equations becomes linear, providing specific boundary condition and applied loads [9]. The method is not material dependent and solely relies on equilibrium, making it ideal for form-finding in conceptual phase of design. Introducing additional constraint makes the problem nonlinear.

Some of the works extended FDM for form-finding of tensegrities [10, 11] in which the sign of force densities can be either negative or positive corresponding to members in compression and tension respectively. In this case, there is also no applied load present and the structure is free-standing for which special considerations should be made [11].

FDM can be used as an interactive tool for form-finding, allowing user to introduce predefined set of q and obtain the corresponding form, exploring possible design solutions [12]. Thrust Network Analysis (TNA) which is based on FDM and graphic statics and its related tool RhinoVAULT [13] have provided such an interactive environment that avoid direct choice of force densities by fixing horizontal projection of the loads and considering only vertical applied loading [14]. Fernández-Ruiz et.al. also developed a tool based on Topological Mapping (TM) to design compression-only structures with inner ribs [15]. Nonetheless, since the force density itself does not represent a physical entity, its effect on the form might be ambiguous, and it does not give explicit control to minimize weight [16] or seek other objectives and impose constraints.

Some of the works that tried to extend FDM method and add some constraints were based on an iterative process that adjust an initial set of force densities and change q so that the solution converges to the point where constraints are almost met. The idea was initially proposed in [9] and then applied, for example, to obtain a constant distribution of forces or stresses throughout the network [9, 17, 18], to impose the magnitude of forces in supports [17], or to enforce the position of loaded nodes [19].

Because of indeterminacy of the network, there are unlimited number of possible solutions that are in equilibrium and transfer loads to the supports. Optimization methods that have been already used for size, shape and topology optimization of the structures can be employed to render it possible to direct FDM in the desired direction and inform the form-finding process. For example, Liew implemented a gradient-based optimization approach to minimize weight and attain target length for each element, with possible constraints on force densities, forces, and node displacements [20]. While most of the works based on FDM that incorporated optimization have used single objective gradient-based algorithms, Descamps et.al. implemented a multi-objective evolutionary strategy for form-finding of a bridge structure [19]. Objectives were mass of the structure and stress in anchorage or length levelling of strut bars. Instead of considering independent force density for each member, they were grouped to decrease the numbers of variables.

This study is concerned with (indeterminate) compression-only networks that may be a gridshell or represent a discretized shell that is in equilibrium with certain boundary condition and applied loads, assuming self-weight to be dominant. An evolutionary optimization algorithm is deployed to minimize an objective function, while satisfying constraints on lengths, forces, and nodal displacements. In this way, desirable criteria can be integrated within form-finding process at the conceptual design phase where there is more freedom to make changes in the design (Fig. 1).

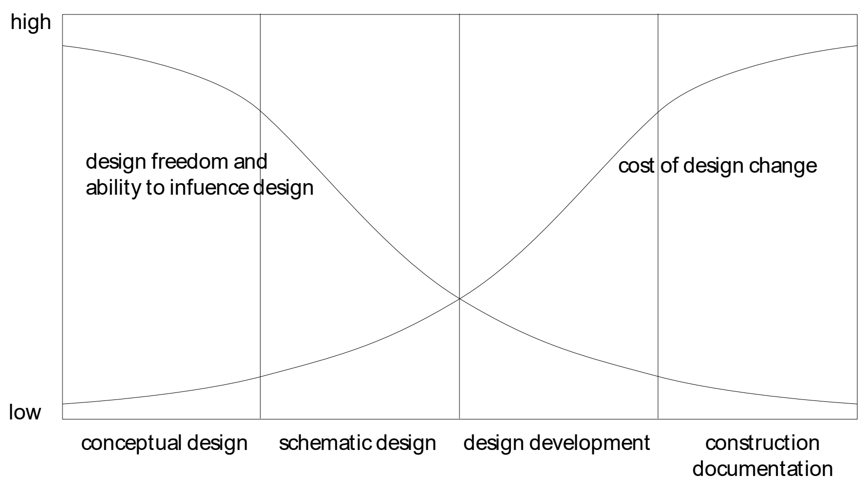


Figure 1. Relationship between design freedom and design knowledge in design process, after [21], [22]

In Section 2, the Force Density Method is explained briefly, before taking a further step to understand the relation between force density and form. Section 3 includes the formulation of the problem and the optimization strategy used in this study. Section 4 provides various examples to illustrate how optimization can inform form-finding process and provide a variety of designs.

2. FORCE DENSITY METHOD

For a given network with m elements and n_s nodes in which n_f nodes are considered to be fixed ($n_s = n + n_f$), the branch-node matrix $\mathbf{C}_s$(or connectivity matrix) that defines the topology of the network is a $(m \times n_s)$ matrix in which the start node of each element is indicated by $+1$ and the end node by -1 with all other arrays equal to zero . It can be divided into two parts $\mathbf{C}_s = [\mathbf{C}\ \mathbf{C}_f]$ with sub-matrixes $\mathbf{C}$ and $\mathbf{C}_f$ corresponding to free nodes and fixed nodes. The equilibrium of nodes that are linearized with introduction of force densities q are [9]:

$$\mathbf{C}^T\mathbf{QCx} + \mathbf{C}_\mathrm{f}^\mathrm{T}\mathbf{QC}_\mathbf{f} = \mathbf{p}_\mathbf{x} \tag{1}$$
$$\mathbf{C}^T\mathbf{QCy} + \mathbf{C}_\mathrm{f}^\mathrm{T}\mathbf{QC}_\mathbf{f} = \mathbf{p}_\mathbf{y} \tag{2}$$
$$\mathbf{C}^T\mathbf{QCz} + \mathbf{C}_\mathrm{f}^\mathrm{T}\mathbf{QC}_\mathbf{f} = \mathbf{p}_\mathbf{z} \tag{3}$$

In which $\mathbf{Q}$ is diagonal matrix of force density of the elements and $\mathbf{p}$ is the vector of applied loads in three principal directions. Summarizing the FDM as the process of finding the position of network nodes for a given boundary condition and a set of applied loads, and considering $\mathbf{D} = \mathbf{C}^T\mathbf{QC}$, the found shape based on equations (1)-(3) can be explained as:

$$\mathbf{x}=\mathbf{D}^{-1}(\mathbf{p}_\mathrm{x}-\mathbf{D}_\mathrm{f}\mathbf{x}_\mathrm{f}) \tag{4}$$
$$\mathbf{y}=\mathbf{D}^{-1}(\mathbf{p}_\mathrm{y}-\mathbf{D}_\mathrm{f}\mathbf{y}_\mathrm{f}) \tag{5}$$
$$\mathbf{z}=\mathbf{D}^{-1}(\mathbf{p}_\mathrm{z}-\mathbf{D}_\mathrm{f}\mathbf{z}_\mathrm{f}) \tag{6}$$

Having found the coordinates from Eq. (4)-(6), the deformed member lengths can be determined. The vector of coordinate differences in three directions are $\mathbf{u} = \mathbf{Cx}, \mathbf{v} = \mathbf{Cy}, \mathbf{w} = \mathbf{Cz}$ of size $(m + 1)$. With $\mathbf{U}, \mathbf{V}, \mathbf{W}$ being the corresponding diagonal matrix of $\mathbf{u}, \mathbf{v}, \mathbf{w}$, the elements lengths can be found as:

$$\mathbf{L} = (\mathbf{U}^2 + \mathbf{V}^2 + \mathbf{W}^2)^{\frac{1}{2}} \tag{7}$$

2.1 Underestanding the force density concept

It has been mentioned that the relation between force densities and equilibrium solution is not clear enough for the user to achieve desired shape [23]. This, however, may be achieved by introducing an optimization problem that facilitates manipulation of force densities in an indirect way to obtain desirable output. Nonetheless, it may be helpful to take a deeper look at how force density works.

For the sake of simplicity, let's consider a very simple structure of Fig. 2. Here a specific load is applied and optimum α is sought to minimize the volume of the structure.

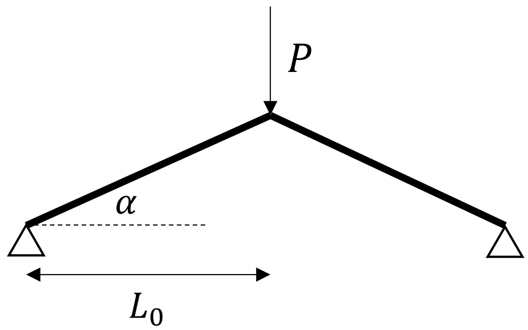


Figure 2. Optimization of a simple structure

The axial forces of the elements (F) can be found as:

$$F = A\sigma_d = \frac{P}{2\sin(\alpha)} \tag{8}$$

And the length of elements are equal to

$$L = \frac{l}{\cos(\alpha)} \tag{6}$$

And volume can be calculated as

$$V = 2AL = \frac{2Pl}{\sigma_d \sin(2\alpha)} \tag{10}$$

Form Eq. (10) it can be understood that for any given set of applied load P and design stress σ_d, the volume is a function of α, the minimum of which happens when $\alpha = {}^{\pi}/_{4}$ that maximizes $\sin(2\alpha)$.

Now let's turn to FDM in which we have direct control over force densities rather than angles. Referring to Eq. (8) and (9), it can be written

$$q = \frac{Force}{Length} = \frac{\frac{P}{2\sin(\alpha)}}{\frac{l}{\cos(\alpha)}} = \frac{P}{2l}\cot(\alpha) \tag{11}$$

Assuming a specific applied load P, setting force density corresponds to selecting a specific angle; the higher the q, the lower the α.

According to the literature, the shape of network found by FDM method is dependent on the relative force density of members, rather than the magnitude of force densities [17]. This, in a sense, is true, but it should be underlined that there is an inverse relation between the magnitude of force densities and height of the network as shown before, even if the relative force density of members remain constant. Moreover, when it comes to the weight of the network, the effect of force density magnitudes becomes prominent, which is different from its effect on the height of structure. For a given network and applied loads, different values for q are given to the elements (identical for all elements) and the results are demonstrated in Fig. 3. It is clear that the shape of networks in plan are identical (concurring with the fact that the relative force density of members control the shape, rather than absolute values, which remained the same here in all different cases), but the height is inversely proportional to the magnitude of q, as interpreted from Eq. (11).

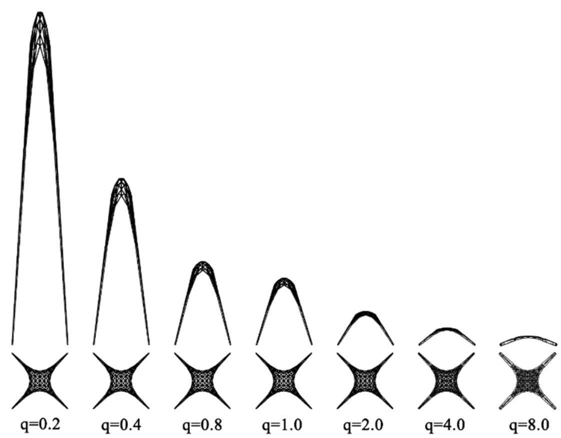


Figure 3. Shape of the structure for constant force densities assigned to all members

If we draw the height and volume of these structures with respect to force density, we can see how force density from $0\ to$ inf corresponds to angle from $\pi/2\ to\ 0$ (Fig. 4). The volume changes are also compatible with Eq. (14) where volume approaches infinite as angle approaches its bounds ($0\ or\ \pi/2$) and the optimum lies somewhere in between.

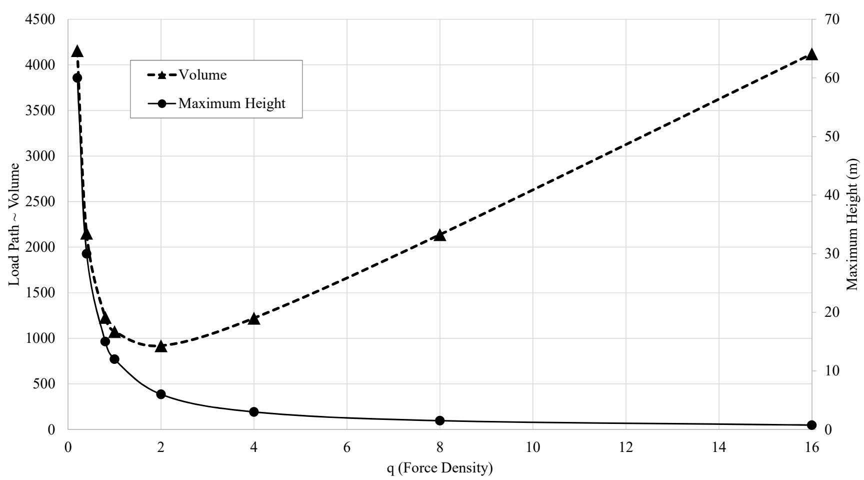


Figure 4. Volume/Load path with respect to different values of force density assigned to all members

The applied load on all nodes in this case was 1. Multiplying it by any arbitrary number (to make it correspond to actual applied loads) and doing the same with force densities would yield the same result, given the invariance of the optimum with respect to magnitude of applied loads; it only depends on relative magnitude of loads. This can also reflect the concept of load path optimization.

3. PROBLEM FORMULATION

Minimizing the volume of a pin-jointed network can be written as

$$min \sum_i V_i = min \sum_i A_i l_i = \frac{1}{\sigma} \sum_i F_i l_i = \frac{1}{\sigma} \sum_i q_i l_i^2 \tag{12}$$

Or in matrix form, Eq. (12) can be explained as:

$$V = \mathbf{q}^{\mathrm{T}} \mathbf{L} \mathbf{l} \tag{13}$$

Which is equivalent to the concept of *load path optimization* proposed in [24, 25]. Here, for specific value of allowable stress σ, weight or material volume can be optimized by minimizing the product of element forces multiplied by their length which itself can be re-written in terms of force densities. Buckling is not considered in this study and allowable stress is taken identical for all members, however, it can be easily extended to consider it too.

3.1 Objective function and constraints

The optimization problem can be written as:

$$\begin{aligned} &\text{minimize } f(\mathbf{q}) = V \\ &\text{s.t.} \begin{cases} g_i(\mathbf{q}) \leq 0 & i = 1, \dots, n_i \\ h_j(\mathbf{q}) = 0 & j = 1, \dots, n_j \\ q_{min} < q < q_{max} \end{cases} \end{aligned} \tag{14}$$

where f is objective function, $\mathbf{q}$ is the vector of member force densities, V is the volume of the structure as described in Eq. (12), and g and h are inequality and equality constraints that may represent any constraint that user may wish to have control over like member lengths, forces/stresses, and nodal displacements. For constraints to be satisfied, the objective function is penalized accordingly. Note that objective function and all the constraints can be defined in terms of force densities. Alternatively, one may wish to have forces as uniform as possible throughout the structure, which is equivalent to having cross sections as similar as possible, considering an identical level of stress in all of them. For this, minimization of weight would not be the priority anymore and similarity of the forces is desired, so the optimisation problem can be defined as minimization of standard deviation of forces in the network:

$$\text{minimize}\, f(\mathbf{q}) = std(\mathbf{f}) \tag{15}$$

where std stands for standard deviation and $\mathbf{f}$ denotes vector of member forces. It should be also highlighted that although the indeterminacy of the network provides a wealth of possible solutions to explore, considering too many constraints could lead to an unfeasible design space.

Given the fact that the network is meant to be completely in compression, the sign of force density for all members should be identical. Either applying downward loads with negative set of force densities or applying upward loads with positive set of force densities are equivalent.

3.2 Optimization method

Genetic Algorithm (GA) is a well-known metaheuristic optimization method that mimic the evolution in nature. It performs a number of operators inspired by nature on a population of possible solutions to generate new individuals, and select fittest ones, successively leading to better performing solutions [26]. In this study, a specific version of GA, Multiple Population Genetic Algorithm (MPGA), is employed. In this implementation, a number of subpopulations perform the standard GA, and after certain number of generations, some individuals *migrate* between them. This strategy can enhance the quality of the results comparing to traditional GA [27]. Further parameters of MPGA are the number of subpopulations, the number of generations required until migration happens, and the topology of the migration. In this study, GA algorithm toolbox developed in Sheffield University [28] has been used which supports MPGA.

4. NUMERICAL EXAMPLES

In a series of examples, it will be shown how optimisation strategy can be employed to inform and help us with form-finding process. The key point lies in how our intentions and goals have to be expressed in terms of an optimisation problem and how different viewpoints can results in a variety of forms. Given the fact that the form-finding is happening in conceptual phase of design, even near-optimum solution could be favorable in that we are exploring forms and not necessarily a single best one.

Defining constraints and determining bounds on variables require attention because too many constraint may narrow down the design space to the point where no feasible solution can be found; loose constraints, on the other hand, may result in large deformation of the network so that the form of the network no longer corresponds to the applied loads with which the form is found. However, it does not mean that not allowing vertices to move in x-y direction is the best practice in terms of consistency with applied loads that represent tributary areas. As shown in Fig. 5, when vertices are fixed in horizontal direction, the tributary areas of the vertices in the found form no longer remain the same; but rather allowing some movement would better result in relatively similar areas and therefore being more consistent with uniform applied loads.

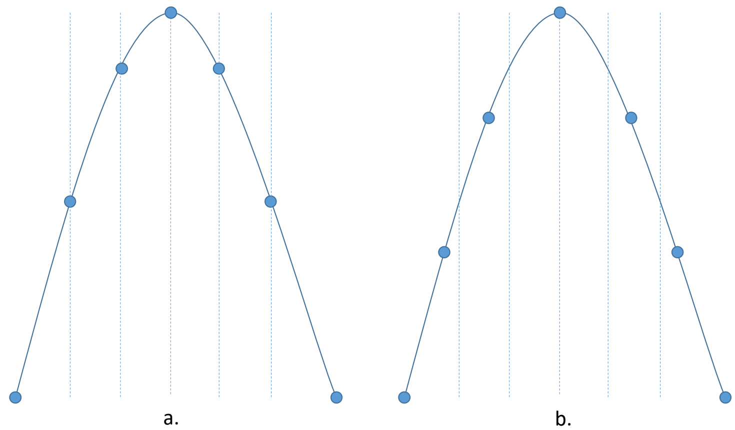


Figure 5. a. fixing vertices along their initial location b. allowing some deviation from initial location

Given the large number of variables present in the optimization process, if there is any number of symmetry in the network, it can be exploited to reduce the number of variables and enhance the speed of algorithm, as well as enforcing the symmetry of the force density within the network. Here symmetry along x and y directions are considered.

A 10x10 network as shown in Fig. 6 is considered with 4 supports at the corners. A uniform vertical applied load is assigned to the vertices representing self-weight of the structure. However, it does not imply that forces have to be necessarily in vertical direction and as shown in section 02, any set of loads in 3 directions can be considered. Different scenarios are introduced for this network to illustrate how form-finding can be informed and directed depending on one's intentions. In order to ensure the structure remains compression-only, the lower bound of force density is set to 0 (the loads are applied upward and force densities are positive).

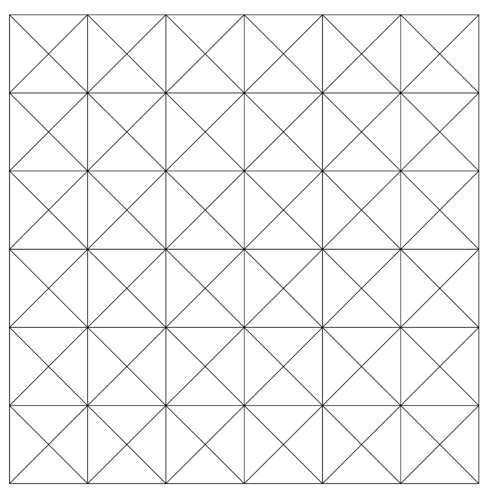


Figure 6. Initial network for form-finding

Example 1

As a starting point, a force density of 1 is applied to all the network and its corresponding volume is taken as 1.0V based on which subsequent forms will be normalized. Then an optimisation is carried out to minimize volume assuming identical force density throughout the network, and the volume decreased to 0.85V with force density of members being equal to 1.797. The results can be seen in Fig. 7, where thickness of lines are proportional to their forces and required cross sections. Both of them have identical shape in plan, while the optimised one has lower height.

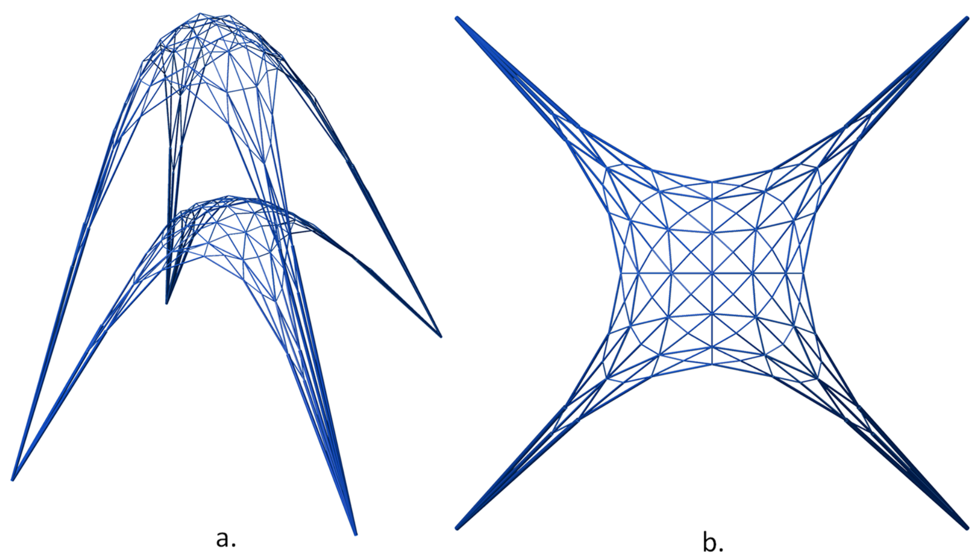


Figure 7. a. 3D view b. plan of forms found for q=1 and q=1.797

Example 2

In this case, independent force densities are considered for each network member in search for forms with lower volume and no other constraint is present other than lower bound on **q**. The result can be seen in Fig. 8 with the volume of 0.13V. As mentioned before, in the absence of constraints some vertices approached each other and even coalesced which is neither consistent with assumed applied loads nor favorable from aesthetic point of view. This requires us to apply sufficient constraint to prevent excessive relocation of vertices.

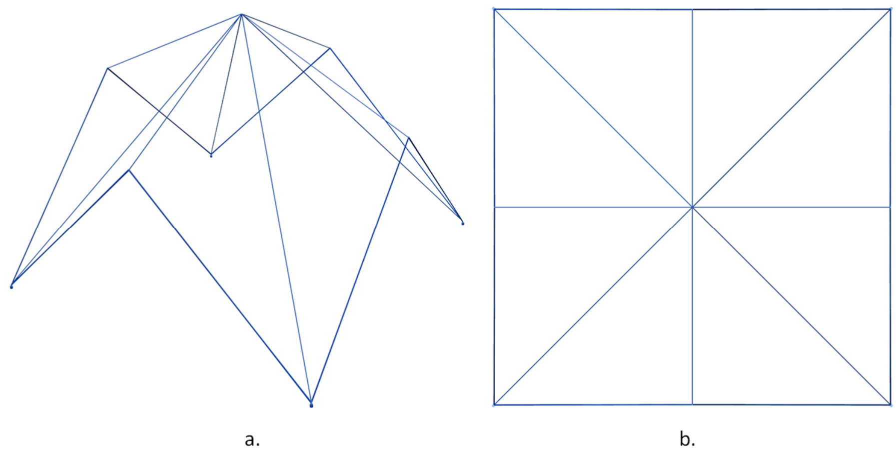


Figure 8. a. 3d view, b. plan of forms found in example 2

Example 3

One way to control excessive deformation of form could be achieved by putting limit on member lengths, here set to be 50% of initial length. Fig. 9 shows the final form with the volume of 0.95V. It can be seen that four main diagonal bands are formed that channel applied forces towards the supports. Although the form is much more consistent with initial grid, some vertices still show relatively large relocations.

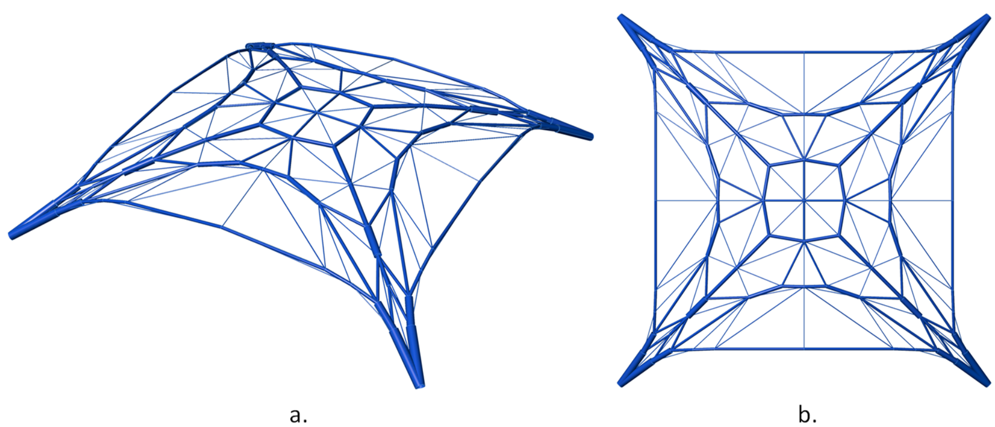
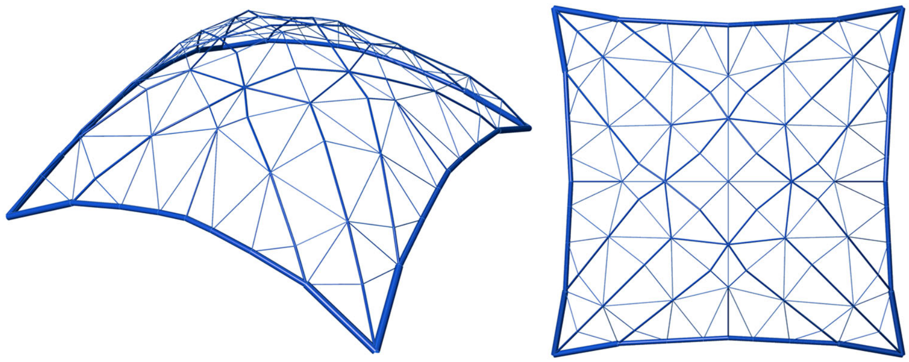


Figure 9. a. 3d view, b. plan of forms found in example 3

Example 4

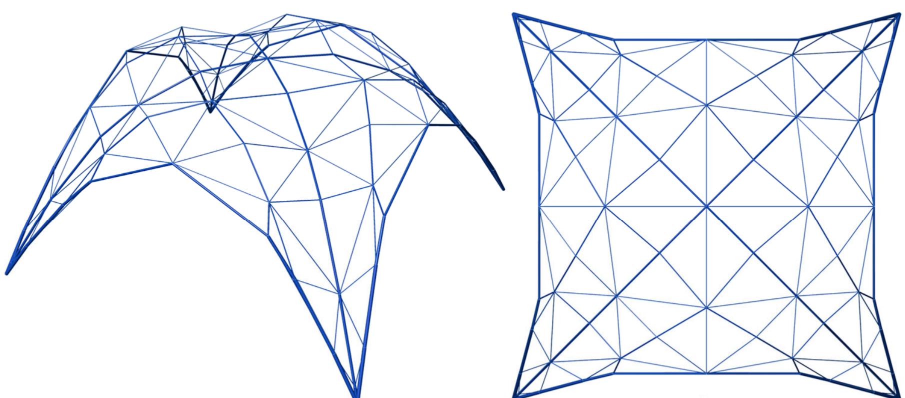
Another approach to restrain network from excessive deformations is to constrain movement of vertices in XY plan. In this example, a maximum of 5% of grid width is assumed as acceptable movement tolerance for vertices. It is clear that the found network has retained its shape in plan, and its volume is almost 1.0V. Periphery arcs and secondary diagonal paths have formed within the network to direct applied forces towards supports.



a. b.

Figure 1- a. 3d view, b. plan of forms found in example 4 with movement tolerance of 5%

With more tolerance on movement of vertices, i.e. 10%, the geometrical density of members in the network changes and members are attracted more towards four support. 3D form also changes considerably and exhibits completely different visual and structural expression.



a. b.

Figure 2- a. 3d view, b. plan of form found in example 4 with movement tolerance of 10%

Example 5

In this case, the objective function is set to be standard deviation of forces of the elements, so the optimisation tries to find forms in which more uniform force distribution happens. It should be noted that 10% movement tolerance for vertices is also applied, because otherwise an irrational form is found where the network coalesces to something like a four-element truss. Fig. 12 shows the best solution with the volume of 0.9V. It can be seen that four primary arches has formed in the periphery of the network and more elements are engaged to direct forces along the network towards the supports.

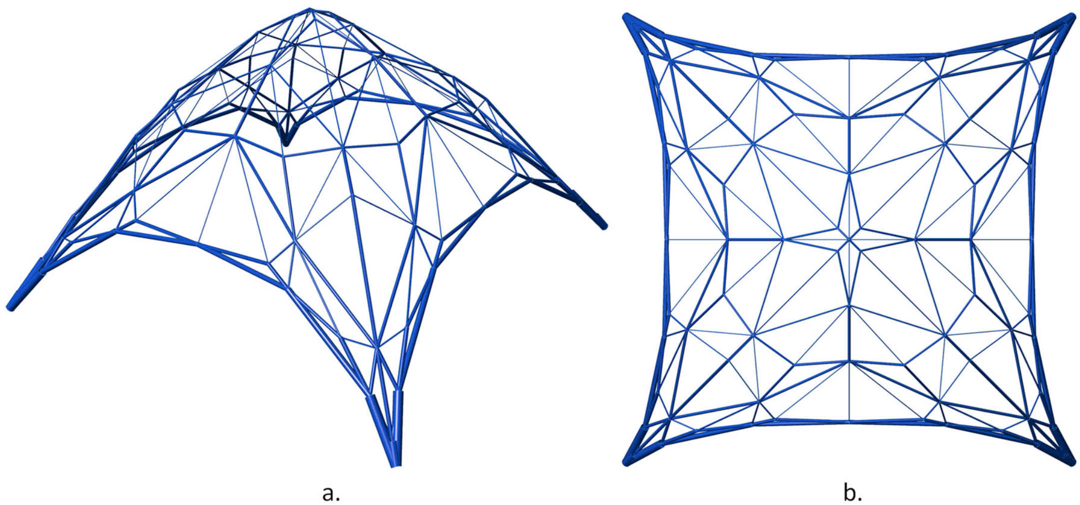


Figure 3- a. 3d view, b. plan of form found in example 5 with movement tolerance of 10%

In order to compare the result of this scenario with that of Example 4, both forms are depicted in Fig. 13 where elements are colored and thickened based on magnitude of force.

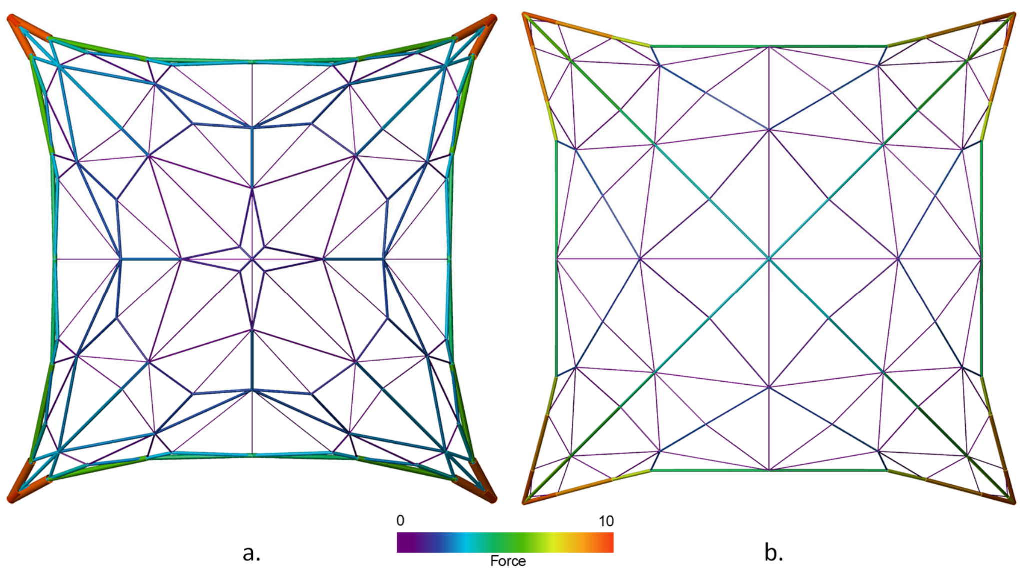


Figure 4. Comparison of the form found in example 4 and 5 in terms of force distribution

5. CONCLUSION

Force density method is a powerful tool for form-finding of structures that is based on equilibrium of forces and therefore can generate efficient structural forms. This method requires a set of force densities for members, apart from boundary condition and applied loads to give its corresponding form in equilibrium. The form-finding process based on FDM is mostly driven by trial and error, without explicit control over forces or geometry of the structure, but rather an overall qualitative evaluation of the form made by user. Optimization can be used to further inform form-finding process and help us explore a variety of forms while minimizing weight and satisfying certain constraints that might be of interest from structural, constructional or aesthetic point of view A framework based on evolutionary optimization and FDM is presented and in a series of examples it has been shown that up to 42%, weight reduction is possible depending on the type and number of constraints applied. It has been also demonstrated how different perspective in defining optimization problem for the same initial network could generate various designs that is desirable from creative design point of view.

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