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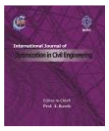

OPTIMAL ANALYSIS OF SKELETAL STRUCTURES VIA FORCE METHOD: A REVIEW

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ABSTRACT

In this paper, a review is provided for the optimal analysis of structures using the graph theoretic force method. An analysis is defined as "optimal" if the corresponding structural matrices (flexibility or stiffness) are sparse, well-structured, and well-conditioned. An expansion process together with the union-intersection theorem is utilized for generating subgraphs, forming a special cycle basis, corresponding to highly localized self equilibration systems. Admissibility checks are used in place of the more common independence checks to speed up the formation of the basis. An efficient solution requires organizing the non-zero entries into various well-defined patterns. Algorithms are provided to form matrices having banded matrices and small profiles. Though the paper considers mainly skeletal structures, the presented concepts are easily extensible to other finite element models. References for such generalizations have been provided. A brief review of swift analysis methods that skirt the harder problem of matrix conditioning is also provided. The iterative nature of optimal structural design via metaheuristic algorithms rewards any speedup in the analysis process. This review recommends utilizing the force method instead of the alternative displacement method to achieve said speedup. The work concludes with a discussion of future challenges in the field of optimal analysis.

Keywords: Optimal Analysis, Sparse Structural Matrices, Well-Structured Matrices, Well-Conditioned Matrices, Graph Theory, Force Method, Optimal Design Using Metaheuristic Algorithms.

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1. INTRODUCTION

The main tools for structural analysis consist of equilibrium of forces, compatibility of displacements and force-displacement relationships. There are mainly two methods for the analysis of structures consisting of the displacement method and the force method.

In the displacement method, one assumes the compatibility to hold and then proceeds to satisfy equilibrium, while in the force method, one assumes the equilibrium to hold and then proceeds to satisfy the compatibility. These methods are known as dual approaches, Argyris and Kelsey [1] and Henderson [2].

In the following diagram (Fig. 1), the inter-relation between analysis and design is illustrated. A structure under external effects, constituting external loads, temperature changes and support settlements is analyzed. Once the internal forces have been calculated, the resulting stresses can be obtained. Subsequently, one carries out the design step for the used materials and the dimensions and/or topology (connectivity) of the structure are modified. This process is repeated until all the design requirements are met.

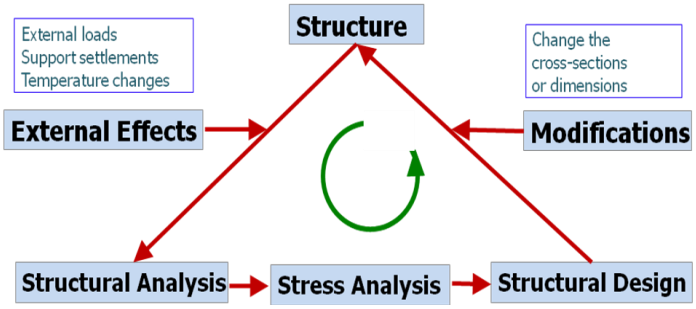

Figure 1: The relationship between structural analysis, stress analysis, and structural design

In the following diagram (Fig. 2) a structure is subjected to loads. Analysis is performed either by the force (flexibility or compatibility) method or the displacement (stiffness or equilibrium) method.

In the force method, first the forces are calculated and then the displacements are obtained (red arrows). While in the displacement method, first the displacements are evaluated and then the internal forces are obtained (blue arrows).

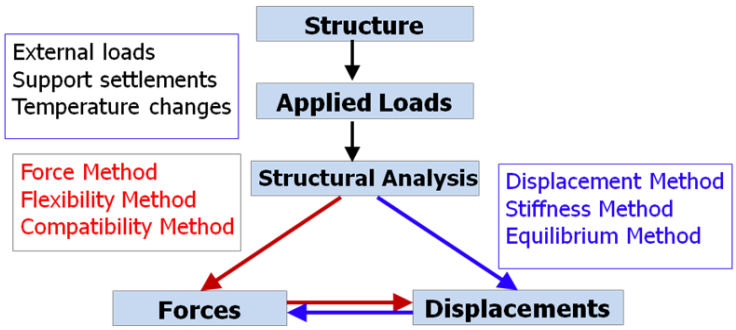

Figure 2: Two dual methods for structural analysis

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After the Introduction section, the paper includes a brief explanation of the optimal analysis in Section 2. The sparsity of structural matrices and its application to skeletal structures, and an efficient method for the force method of frame structures are explained in Section 3. Nodal ordering with the goal of producing well-structured matrices, bandwidth reduction and its extension to profile reduction are the focus of Section 4. The subsequent Section 5 proposes a method for selecting well-conditioned structural matrices. Though the method is used for the force method, it can easily be extended to the displacement approach. Swift analysis of symmetric and regular structures is briefly presented in Section 6, followed by a discussion on the optimal analysis for optimal design in Section 7. Here, the force method is suggested for structures having a lower static degrees of indeterminacy compared to their kinematic degree of indeterminacy. A compilation of our conclusions is presented in the final section. References for basic necessary concepts and definitions are included in Appendix A, followed by the proof of the "union-intersection theorem" provided in Appendix B.

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2. OPTIMAL ANALYSIS OF STRUCTURES

If the structural matrices (flexibility or stiffness), resulting from the analysis of a structure, have the following properties:

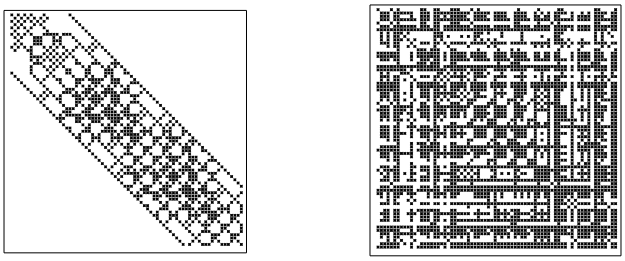
- Sparse (having a high percent of zero entries),
- Well-structured (non-zero entries are positioned in a special pattern),
- Well-conditioned (having small off-diagonal terms compared to diagonal terms),

then, the analysis will be considered an *Optimal Analysis.*

Though the above properties have been separately studied by many researchers, the term "optimal analysis" is defined in Kaveh [3] and the collective study of the above properties is published in Refs. [4-6], and books [7-9]. It should be noted that this term is quite different from "optimal design of structures", where mathematical programming and/or metaheuristic algorithms are utilized.

2.1. Sparsity

Sparsity is an important and well-established subject in the fields of mathematics, science and engineering. In mathematics, the sparsity of a matrix is usually given, and the goal is to solve the corresponding equations efficiently, Tewarson [10], Pissanetzky [11], and Duff et al. [12]. Conversely, in engineering, the problems are solved such that the resulting structural matrices become sparse, i.e. having a lower number of non-zero entries Henderson [2], Henderson and Maunder [13], Maunder [14], Kaveh [4, 7], Cassell et al. [15]. Typical sparse and non-sparse matrices corresponding to the analysis of identical frame structure are illustrated in Fig. 3.

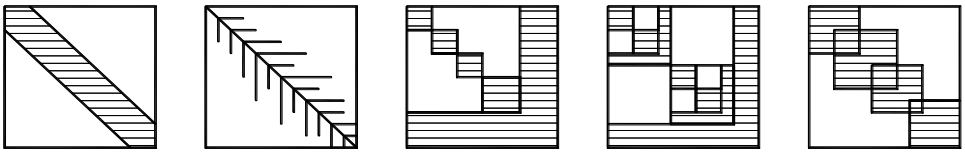


(a) Sparse matrix **(b) Non-sparse matrix**

Figure 3: Examples of a sparse matrix and a non-sparse matrix.

2.2. Well-structured Matrices

Once the number of entries of a matrix is reduced, an efficient solution requires organizing the non-zero entries into various well-defined patterns. Depending on the method to be used for the solution, these patterns should be different. The number of such patterns can be extensive and here only some well-known patterns utilized in Gaussian elimination, skyline method, substructuring, and frontal method are shown (Fig. 4). Methods for patterning structural matrices are developed by Kaveh [4-7] and others.



(a) Banded form (b) Profile form (c) Partitioned form (d) Nested form (e) Frontal form

Figure 4: Different patterns of well-structured matrices for the solution of sparse matrices

There are many other well-structured matrices such as three-diagonal matrices, five-diagonal matrices, block diagonal matrices, block diagonal matrices with corner blocks, which require special ordering and often occur in symmetric, cyclically symmetric and regular structures, Kaveh [8].

2.3. Well-conditioned Matrices

Linear algebra and matrix algebra are the main tools for improving the conditioning of matrices. Methods for this purpose are developed by Shah [16], Goodspeed and Martin [17] who employed the substructuring technique for improving the conditioning of stiffness matrices. Kaveh [18] and Cassell [19] developed graph theoretic methods for improving the conditioning of flexibility matrices and Kaveh and Ghaderi [20] presented efficient methods for improving the conditioning of the stiffness matrices of skeletal structures. A typical well-conditioned matrix having dominant diagonal entries is shown in Fig. 5.

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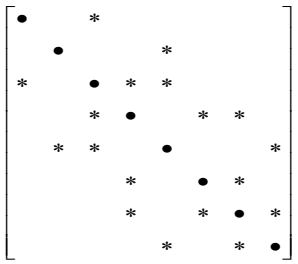

Figure 5: A well-conditioned symmetric matrix with dominant diagonal entries

In subsequent sections, the progress made in providing the three aforementioned properties, required for optimal analysis, are discussed in detail.

3. FORMATION OF SPARSE STRUCTURAL MATRICES

The flexibility and stiffness matrices are pattern-equivalent to cycle adjacency and cut set adjacency matrices, respectively. While there is a special cut set basis known as a co-cycle basis which can be used for generating equilibrium equations around the structural nodes, such a suitable simple cycle basis is not available in the force method. This is why this section will be devoted to the formation of cycle basis corresponding to sparse flexibility matrices.

The "generalized cycle basis" has been developed for the analysis of skeletal structures (planar trusses, space trusses, planar frames and space frames) using the force method, Kaveh [3,5,7]. For frame structures suitable cycle basis is due to Henderson [2]. Henderson and Maunder [12] developed valuable topological concepts for the manual selection of cycle bases leading to sparse flexibility matrices for frame structures. Kaveh made extensive contributions to computer methods for the formation of subminimal cycle bases [21-23]. An efficient program can be found in Kaveh [24]. This idea has been extended to the formation of suboptimal cycle bases in [25]. In this section an algorithm is provided using these concepts [4].

3.1. Generalized Cycle Bases of a Graph

In this section, S is considered to be the graph model of a skeletal structure. For the function $\gamma(S) = aM(S) + bN(S) + c\gamma_0(S)$ associated with the corresponding structure, the coefficients b and c are assumed to be integer multiples of the positive coefficient a, with M(S), N(S) and $\gamma_0(S)$ being the number of members, nodes and components of the structural model. Here, only those coefficients provided in Table 1 are of interest.

More general functions applicable to degrees of static and kinematic indeterminacies are also defined for skeletal structures and finite element models [3,4].

3.1.1. Definitions

In this section, some concepts of graph theory are generalized in order to unify the optimal force method analysis for planar and space trusses and planar and space frames.

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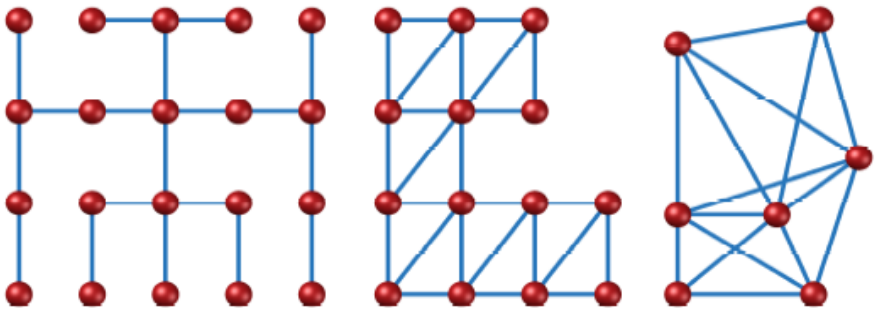
Table 1: Coefficients of γ(S) for different types of skeletal structures.

Type of structure	a	b	c
Plane truss	+1	–2	+3
Space truss	+1	–3	+6
Plane frame	+3	–3	+3
Space frame	+6	–6	+6

Definition 1: An *elementary subgraph* is defined as a subgraph which does not contain any subgraph $S_i' \subseteq S_i$ with $\gamma(S_i') > 0$. A connected subgraph T of S containing all the nodes of S is called a *spanning γ-tree* if $\gamma(T) = 0$ and it is rigid. The word "rigid" is important and essential for the definition to be true. For $\gamma(S_i) = b_1(S_i)$, a γ-tree changes to a tree as defined in graph theory.

A structural model as a γ-tree, becomes statically determinate when γ(S) describes the degree of static indeterminacy (DSI) of the structure. The ensuing stress resultants can uniquely be obtained everywhere in the structure by pure equilibrium. Three γ-trees corresponding to different functions are shown in Fig. 6.



(a) γ(S)=3M–3N+3. **(b) γ(S)=M–2N+3.** **(c) γ(S)=M–3N+6.**

Figure 6: Examples of different γ-trees.

It should be mentioned that $\gamma(T) = 0$ does not guarantee the rigidity of a γ-tree. For example, both graph models illustrated in Fig. 7 satisfy the condition $\gamma(T) = 0$; however, none of these trusses are rigid. Rigidity is extensively studied by graph theoretic methods by mathematicians and the interested reader can refer to Refs. [26-28]. However, only methods for controlling the rigidity of planar trusses have been developed by Lovász and Yemini [29] and Sugihara [30]. For space trusses with special configuration only some partial results are available,

Definition 2: A member of S − T is called a γ-chord of T. The collection of all the γ-chords of a γ-tree is known as the γ-cotree of S.

Definition 3: A removable subgraph Sj of a graph Si, is an elementary subgraph for which $\gamma(Si - Sj) = \gamma(Si)$, i.e. the removal of Sj from Si does not alter its degree of static indeterminacy. A γ-tree of S containing two specified nodes, that has no removable subgraph is called a γ-path between these two nodes.

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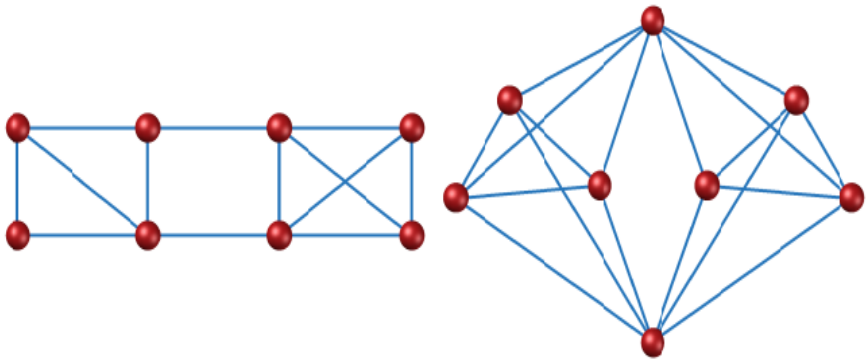
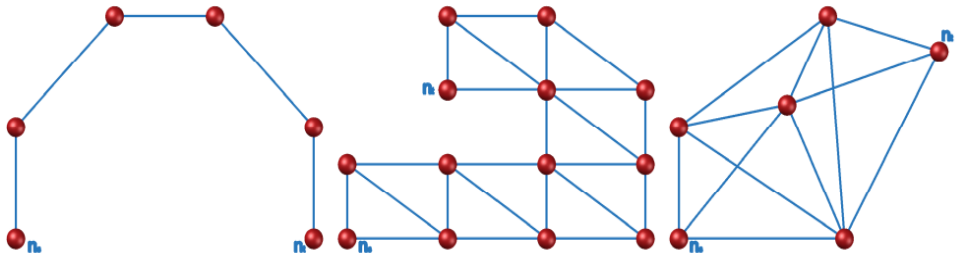
As an example, the graphs shown in Fig. 8 are γ-paths between the two specified nodes ns and nt.



(a) γ(S)=M–2N+3. **(b) γ(S)=M–3N+6.**

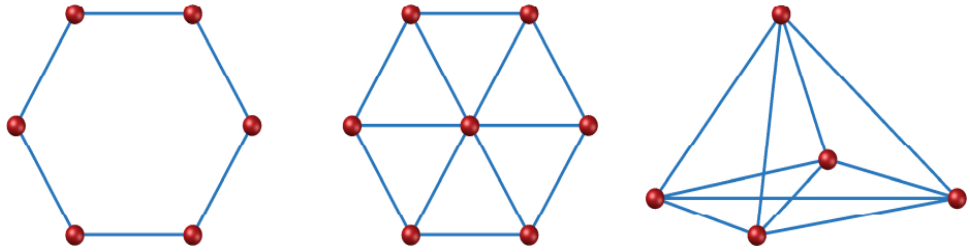
Figure 7: Structures satisfying γ(T) = 0 which are not rigid.



(a) γ(S)=α(M–N+1). **(b) γ(S)=M–2N+3.** **(c) γ(S)=M–3N+6.**

Figure 8: Examples of γ-paths between two specified nodes n_s and n_t.

Definition 4: A connected rigid subgraph C_k of S with $\gamma(C_k) = a$, containing no removable subgraph is called a γ-cycle of S. The number of members of C_k, denoted by $L(C_k)$, is called the length of C_k. Examples of γ-cycles are illustrated in Fig. 9.



(a) γ(S)=α(M–N+1). **(b) γ(S)=M–2N+3.** **(c) γ(S)=M–3N+6.**

Figure 9: Examples of γ-cycles considering different functions.

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A γ-cycle is also defined as the underlying subgraph on which S.E.Ss can be constructed. For trusses one S.E.S. can be formed on a γ-cycle, while for the frame structures three or six S.E.Ss can be constructed on each γ-cycle, respectively. This definition is compatible with the above definition, however, one might be able to generate a γ-cycle for trusses using the later definition.

Definition 5: For m_i being a γ-chord of T, the $T \cup m_i$ contains a γ-cycle C_i which is defined as a *fundamental γ-cycle* of S with respect to T. Using the Union-Intersection Theorem provided in Appendix B, it can easily be shown that

$$\gamma(T \cup m_i) = 0 + (a+2b+c) - (2b+c) = a,$$

indicating the existence of a γ-cycle in $T \cup m_i$. For a rigid T, the corresponding fundamental γ-cycle is also rigid, since the addition of an extra member between the existing nodes of a graph cannot destroy the rigidity. A fundamental γ-cycle can be obtained by omitting all the removable subgraphs of $T \cup m_i$.

Definition 6: A maximal set of independent γ-cycles of S is defined as a *generalized cycle basis* (GCB) of S. Similarly, a maximal set of independent fundamental γ-cycles is called a *fundamental generalized cycle basis* of S. This basis will have dimension as $\eta(S) = \gamma(S)/a$.

For the following, example, a generalized cycle basis of a planar truss is illustrated in Fig. 10.

Definition 7: A *generalized cycle basis-member incidence matrix* **C** is an $\eta(S)\times M$ matrix having entries as 0 and 1, where $c_{ij} = 1$ if γ-cycle C_i contains the member m_j, and $c_{ij} = 0$ otherwise. The *generalized cycle adjacency matrix* is defined as **D** which is an $\eta(S)\times\eta(S)$ matrix.

3.1.2. Minimal and optimal generalized cycle bases

A generalized cycle basis $C = \{C_1, C_2, ..., C_{\eta(S)}\}$ is called *minimal* if it corresponds to a minimum value of:

$$L(C) = \sum_{i=1}^{\eta(S)} L(C_i). \tag{1}$$

The sparsity of matrix **C**, denoted by $\chi(\mathbf{C})$ is equal to the total number of selected cycle set, C denoted by L(C), A minimal GCB can be defined as a basis which corresponds to minimum number of non-zero entries of the matrix **C**. A GCB for which L(C) is near minimum is called a *subminimal* GCB of S.

A GCB corresponding to maximal sparsity of the GCB adjacency matrix is called an *optimal* generalized cycle basis of S. If $\chi(\mathbf{CC}^t)$ does not differ considerably from its minimum value, then the corresponding basis is termed *suboptimal*.

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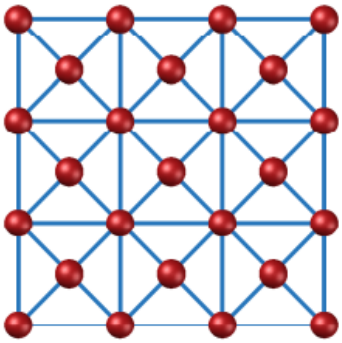
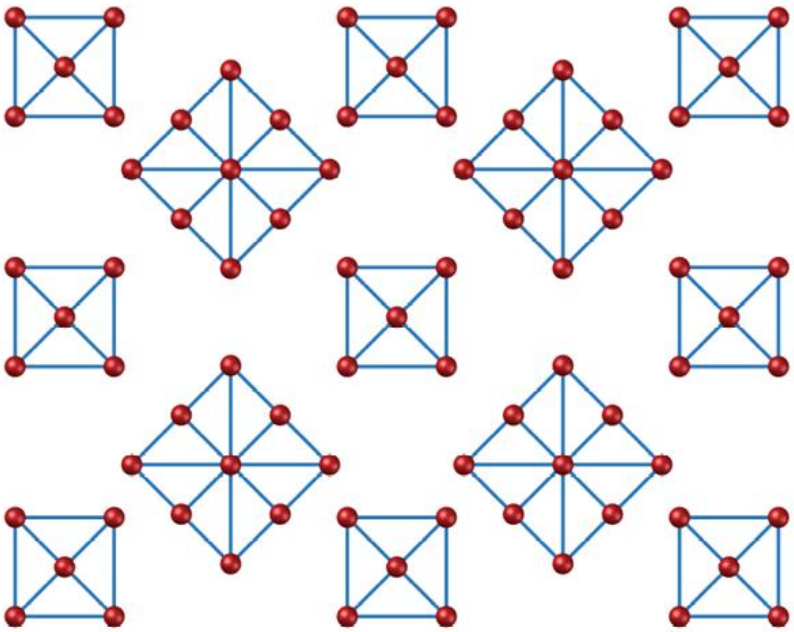


(a) A planar truss S.



(b) A generalized cycle basis of S.

Figure 10: A planar truss S, and the elements of a GCB of S.

The matrix intersection coefficient $\sigma_i(\mathbf{C})$ of row i of the GCB incidence matrix **C** is the number of row j such that:

(a) $j \in \{i+1, i+2, \ldots, \eta(S)\}$,

(b) $C_i \cap C_j \neq \varnothing$, i.e. there is at least one k such that the column k of both γ-cycles C_i and C_j (rows i and j) contain non-zero entries.

Now it can be shown that:

$$\chi(\mathbf{C}\mathbf{C}^t) = \eta(S) + 2 \sum_{i=1}^{\eta(S)-1} \sigma_j(C). \tag{2}$$

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This equation shows the correspondence of a GCB incidence matrix **C** and that of its GCB adjacency matrix. In order to minimize $\chi(\mathbf{C}\mathbf{C}^t)$, the value of $\sum_{i=1}^{\eta(S)-1} \sigma_j(C)$ should be minimized, since $\eta(S)$ is a constant for a given structure S, i.e. γ-cycles with a minimum number of overlaps should be selected.

Different algorithms are developed for the formation GCB, however, there are still computational problems which should be resolved [4,5]. Forming a γ-cycle for a general function in not an elementary problem, and choosing admissible γ-cycles is not an easy task. For a cycle basis, the problem becomes simplified as discussed in the following sections [4,5].

3.2. An Efficient Algorithm for the Formation of Subminimal Cycle Basis

A simple expansion process has been used in structural analysis for the formation of simple trusses by Muller Breslau [31]. Kaveh generalized the idea to general skeletal structures [3] and finite element models [4]. In this expansion, the added subgraphs could be cycles, cut-sets, co-cycles, γ-cycles, or the subgraph model of substructures.

3.2.1. Expansion for the formation of a cycle basis; graph theoretic method

Consider an expansion process as,

$$C_1 = C^1 \rightarrow C^2 \rightarrow C^3 \rightarrow \ldots \rightarrow C^{b_1(S)} = S \tag{3}$$

where $b_1(S)$ is the first Betti number of S. A cycle C_{k+1} is called an admissible cycle, if

$$b_1(C^{k+1}) = b_1(C^k \cup C_{k+1}) = b_1(C^k) + 1. \tag{4}$$

A cycle basis can be generated by the above expansion process.

It should be noted that in an expansion process an admissible cycle is independent of the previously selected cycles, but an independent cycle is not necessarily admissible.

3.2.2. Minimal cycle on a member

Formation of a shortest route tree: Two shortest route trees (SRTs) are generated from O and O' in Fig.11. The contours are illustrated in dashed lines, see Appendix A.

In order to generate a shortest length cycle C_i on a member m_j, two SRTs are formed from the two end-nodes n_s and n_t. Soon these trees intersect at node n_c. The backtracking to n_s and n_t is then performed. These two paths, together with m_j form a shortest cycle on m_j. One can also use a single SRT routed from n_s (excluding m_j), such that the generated subtree includes n_t. Again, backtracking is performed and the generated path together with m_j results in the formation of the shortest cycle on m_j, Fig. 12.

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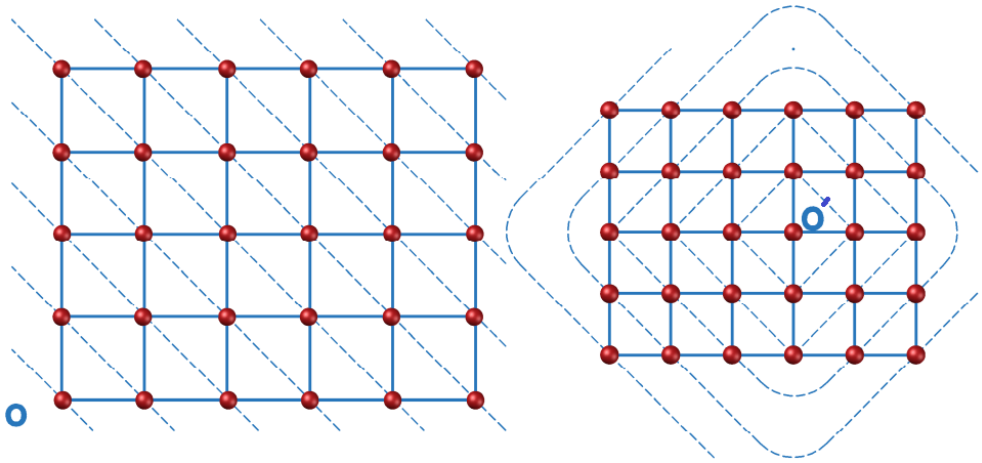

Figure 11: Two shortest route trees generated from O and O'

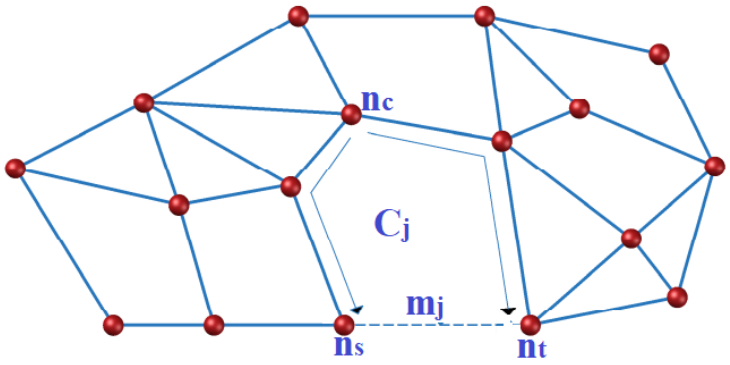

Figure 12: Generation of a minimal length cycle on a member m_j

3.2.3. An example of selecting admissible cycles as a basis by expansion

A simple graph S with $b_1(S) = 5$ is considered and a selection of 5 admissible cycles is performed as shown in Fig. 13. First, three 3-sided cycles are generated, followed by a 4-sided cycle. Then another 3-sided cycle is formed.

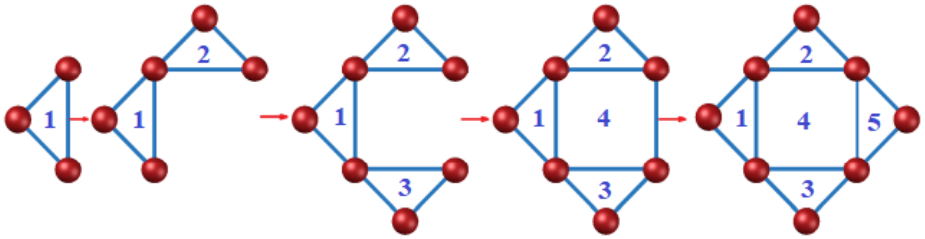

Figure 13: Formation of a cycle basis using an expansion process

After discussing the mathematical concepts which is involved in the generation of a subminimal cycle basis, it should be mentioned that different algorithms have been

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developed by Kaveh [4-7] for the formation of a subminimal cycle basis. The following algorithm is the simplest and most informative algorithm.

3.3. Algorithm (Kaveh 1974)

- Step 1: Select a pseudo-centre node of maximal degree O. Such a node can be selected manually or automatically utilizing the graph or algebraic graph theoretical methods [3-5].

- Step 2: An SRT rooted at O can then be generated obtaining the set of its chords that should be ordered according to their distance from O.

- Step 3: One minimal cycle on each chord should be formed in turn. A corresponding simple path is chosen which contains members of the tree and the previously employed chords. This provids the admissibility of the selected cycle.

This method selects subminimal cycle bases, using the chords of an SRT. The nodes and members of the tree, and hence the cycles, are partially ordered according to their distance from O.

Comparing this algorithm with simple algorithm of Kirchhoff [32] reveals the following improvements on the formation of fundamental cycle basis of a graph:

1. Starting from the center of the graph reduces the length of the fundamental cycle compared to using an arbitrary spanning tree.

2. Ordering the chords increases the sparsity of the cycle adjacency matrix and also helps to make well-structured flexibility matrices. This is an interesting situation where the sparsity and well-structuring are performed simultaneously.

3. Adding the used chords to tree member allows the turn back and reduces the length of the selected cycle [21]. This method is the graph theoretic turn back method, that is extended to algebraic force method by Topcu [33], and Kaneko et al. [34], and improved by Topcu and Soyer [35].

4. ORDERING FOR WELL-STRUCTURED MATRICES

After sparsity has been achieved for the structural matrices, the non-zero entries should be generated such that the corresponding matrix has a suitable pattern. For example, the matrix in question can be made banded for Gaussian elimination, the profile can be minimized for the skyline solution, or the front-width can be minimized for the frontal solution. Of course, there are other suitable patterns for symmetric and regular structures. For the force method, the selected cycles are ordered to obtain banded flexibility matrices.

The first direct method for bandwidth reduction was recognized by the graph theorist Frank Harary in 1967. He posed the following question:

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For a graph S with N(S) nodes, how can labels 1, 2, ..., N(S) be assigned to the nodes such that the maximum absolute value of the difference between the labels of all pairs of the adjacent nodes is minimized? In the following we will address this question in detail.

4.1. Early Developments

Cuthill and McKee [36] developed the first graph-theoretical method for reducing the bandwidth of stiffness matrices. In their work, a level structure was used which was called the "spanning tree" of a structure. Kaveh's interest in bandwidth reduction was motivated by his interest in generating and ordering the elements of cycle bases, in order to reduce the bandwidth of the flexibility matrices. For this purpose a *shortest route tree* (SRT) was utilized. The application of this method has been extended to the elements of a cutset basis in order to reduce the bandwidth of stiffness matrices. It was subsequently observed that there is a close relation between Cuthill-McKee´s *level structure* and the Kaveh´s SRT [37-39]. Multiple shortest route trees are utilized for bandwidth reduction [40], and a connectivity coordinate system is defined for performing nodal ordering [41]. However, there is a difference between these two types of trees in that an SRT is an optimizer and contains additional information about the connectivity properties of the corresponding structure.

A matrix **A** is called *banded*, if all its non-zero entries are confined within a band, formed by diagonals parallel to the main diagonal. Thus, $A_{ij} = 0$ when $|i - j| > b$, and $A_{k,k-b} \neq 0$ or $A_{k,k+b} \neq 0$ for at least one value of k. Here, b is the half-bandwidth and 2b+1 is known as the *bandwidth* of **A**. These two values are alternately called the bandwidth of the matrix **A**.

4.2. Shortest Route Tree

An SRT of S rooted at O, as shown in Fig. 14(a), denoted by SRT_O, has the following characteristics:
$w(C_0) = 1$, $w(C_1) = 2$, $w(C_2) = 3$, $w(C_3) = 4$, $w(C_4) = \mathbf{5}$, $w(C_5) = 5$, $w(C_6) = 4$, $w(C_7) = 3$, $w(C_8) = 2$ and $w(C_9) = 1$. Hence $h(SRT_O) = 9$ and $w(SRT_O) = 5$.

For the same graph, an SRT rooted at O´, as shown in Fig. 14(b), leads to $h(SRT_{O'}) = 5$ and $w(SRT_{O'}) = 9$.

An SRT partitions the node set of S into subsets according to their distance from the root. Each subset is known a *contour* (or level) of the SRT, denoted by C_i. The contours of an SRT have the following properties:

$$\begin{aligned} &\text{Adj } (C_i) \subseteq C_{i-1} \cup C_{i+1}, \qquad 1 < i < m \\ &\text{Adj } (C_1) \subseteq C_2, \\ &\text{Adj } (C_m) \subseteq C_{m-1}. \end{aligned} \tag{5}$$

The number of nodes in each contour is called the *width* of that contour. The largest width of the contours of an SRT is called the *width of the SRT* rooted at the starting node O, denoted by $w(SRT_O)$. This number is known as the *width number* of SRT rooted from O. The number of contours of an SRT (except the starting node contour) is the *height* of the tree

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denoted by h(SRT_O). The *longest* SRT is the one having maximal height and the *narrowest* SRT is the one which corresponds to minimal width.

4.3. Nodal Numbering

Once the starting node and the corresponding SRT is generated, back track from a node of minimum valency and generate a path known as the *transversal* containing N_1, N_2, N_3, …, N_k.

Step 1: Number N_1 as "1".

Step 2: N_2 is assigned as number "2" and an SR subtree is generated from N_2, numbering the nodes of C_2 in the order of their occurrence in this SR subtree.

Step 3: The process of Step 2 is repeated for numbering the nodes of C_3, C_4, ... , C_m, sequentially using N_3, N_4, ... , N_m as the starting nodes of SR subtrees, until all the nodes of S are numbered.

Now the numbering can be reversed, in a way similar to that of the Reverse Cuthill-McKee algorithm, for possible reduction of fill-ins in the process of Gaussian elimination.

4.4. Kaveh's 4-step Algorithm

The following four-step algorithm was developed by Kaveh for the nodal ordering of graphs leading to banded node adjacency matrices. This method can directly be used for nodal ordering of skeletal structures resulting in banded stiffness matrices.

1. Finding a suitable starting node;
2. Decomposing the node set of S into ordered subsets (contours);
3. Selecting a connected path (transversal) containing one representative node from each contour;
4. Ordering the nodes within each contour, sequentially, to obtain the final nodal numbering of S.

A graph model is considered as shown in Fig. 14(a). First a good starting node A is found, and the corresponding SRTs are depicted in Fig. 14(b). A transversal is selected as shown in bold lines, Fig. 14(c). Then nodes are numbered contour by contour, employing the representative nodes as the starting nodes of SR subtrees Fig. 14(d).

4.5. Ordering for Profile Reduction

In order to proceed with main algorithms for profile reduction, some definitions will now be stated in the following:

The profile of an n×n square matrix **A** is defined as,

$$P = \sum_{i=1}^{N} b_i, \tag{6}$$

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where the row bandwidth, b_i, for row i is defined as the number of inclusive entries from the first non-zero element in the row to the (i+1)-th entry.

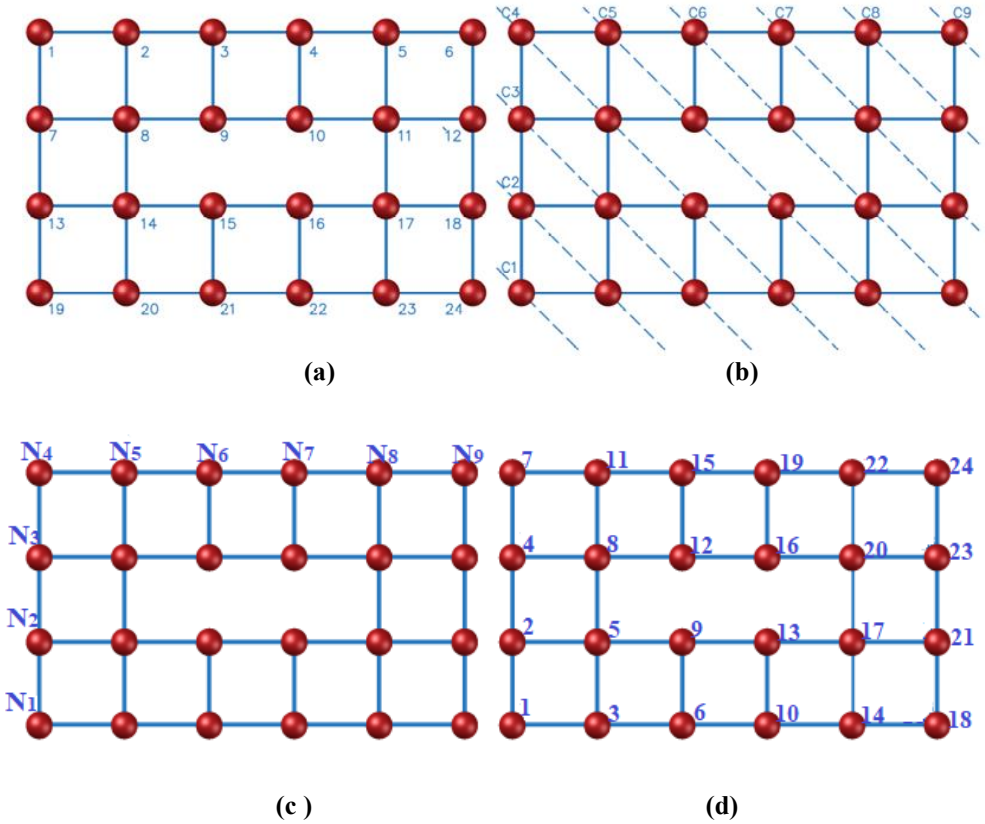

Figure 14: Steps of ordering showing the selected transversal

The numbering and control of nodes in the priority queue are carried out through the assignment of status, based on the numbering strategy of King [42]

The nodes in the King's strategy can be categorized more formally as follows:

Prior to the numbering all the nodes of a graph model G of the considered FEM are assigned *inactive status*. When a node of G is inserted in the priority queue, it is assigned *preactive status*. After a node is numbered, it is assigned *postactive status*. Nodes which are adjacent to a postactive node and do not have postactive status are defined as having *active status*. King´s algorithm has been generalized by Sloan through introducing a priority queue to control the order to be followed in the numbering of the nodes, Fig. 15. This algorithm consists of the following two phases:

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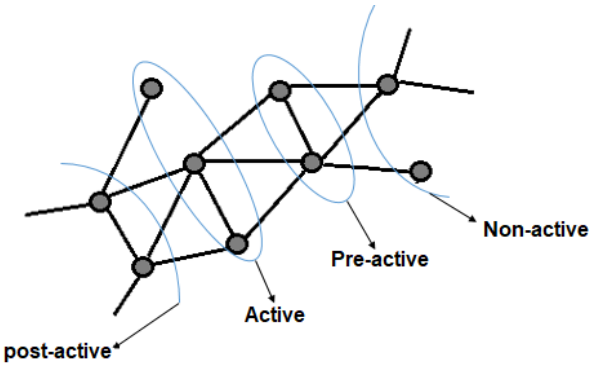

Figure 15: Definition used for Sloan's algorithm

The priority function in Sloan's method [43,44] is generalized by Kaveh and Rahimi Bonderabady [45] considering additional vectors known as vectors of the graph parameters, Kaveh [46].

$$\bar{\phi} = \sum_{i=1}^{p} w_i \mathbf{v}_i, \tag{8}$$

where $\bar{\phi}$ is the priority function, $\mathbf{v_i}$ (i=1, … , p) are the normalized Ritz vectors representing the graph parameters, and $\mathbf{w_i}$ (i=1, … , p) are the coefficients of the Ritz vectors (Ritz coordinates) which are unknowns, and p is the number of parameters being employed. In this paper we set p=5 and v_1 - v_5 vectors are selected as:

v_1 contains the degrees of the nodes,
v_2 comprises of the 1-weighted degrees of the nodes, and
v_3 and v_4 are distances of the nodes from two pseudo-peripheral nodes, and
v_5 contains the 2-weighted degrees of the nodes of the graph [45].

The k-weighted degree of a node n_i is defined as the sum of the degrees of all nodes in a distance k from n_i. Here, k=1 and 2 are used, however, higher values of k may also be considered.

The ideas presented for nodal number are general and are easily extended to finite element nodal and element ordering, cycle and γ–cycle ordering, and substructure ordering [47-51].

5. FORMATION OF WELL-CONDITIONED STRUCTURAL MATRICES

Shah studied the ill-conditioning of stiffness matrices for the displacement method, where he suggested methods for improving the conditioning of stiffness matrices. A mathematical investigation of matrix error analysis is that of the Rosanoff and Ginsburg. In their work, it was shown that numerically unstable equations may arise in physically stable problems. Thus, the need for the repeated recalculation of matrix conditioning numbers associated with

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various patterns of formulation is emphasized. The effect of substructuring on the conditioning of stiffness matrices, was investigated by Grooms and Rowe [52], who found out that substructuring does not significantly influence the solution accuracy for ill-conditioned systems. Filho [53] used an orthogonalization approach for the best conditioning of flexibility and stiffness matrices; however, this can be an impractical approach for large and complex structures.

5.1. Condition Numbers

For measuring the conditioning of a matrix, different numbers (norms) are available inliterature. In the following, three simple conditioning numbers utilized in structures are provided.

5.1.1. Condition Number 1: Ratio of largest and smallest eigenvalues

A common condition number for matrices is $| \lambda_{max} | / | \lambda_{min} |$, with λ_{max} being the eigenvalue of the largest modulus and λ_{min} is the eigenvalue of the least modulus as defined in the following (Rosanoff and Ginsburg [54]):

The ratio of the extreme eigenvalues of a matrix $| \lambda_{max} | / | \lambda_{min} |$ can be taken as its condition number. It can also be shown that, the logarithm to the base ten of this condition number, is roughly proportional to the maximum number of significant figures lost in inversion or in the solution of simultaneous equations. Therefore, the number of good digits in the solution, g, is evaluated by:

$$g = p - \log(| \lambda_{max} | / | \lambda_{min} |) = p - PL \tag{9}$$

5.1.2. Condition Number 2: The ratio of determinants

Since the best conditioned matrix for inversion is a diagonal one, the following parameter may also be adopted as a practical approach to use for the conditioning of a matrix **A**. Define

$$\varepsilon = \det [\mathbf{A}] - \det [\mathbf{A}_{ii}], \tag{11}$$

where $[\mathbf{A}_{ii}]$ is a diagonal matrix consisting of the diagonal entries of [**A**] and “det” is used for determinant.

The value of ε approaches zero for an ideally conditioned matrix. Therefore, the following condition number, PDET, can be employed:

$$PDET = \det [\mathbf{A}] / \det [\mathbf{A}_{ii}] \tag{12}$$

PDET should approach unity, for an ideally conditioned matrix. This condition number is simple and very easy to calculate.

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5.1.3. Condition Number 3: Determinant of a row-normalized matrix

A simple and workable measure of conditioning of a set of equations is to evaluate the determinant of the row-normalized matrix of the coefficients of the set. The determinant of the row-normalized **A**, denoted by PN, is a good measure for the conditioning of **A**. Obviously, the magnitude of this determinant lies in the range,

$$0 < PN \leq 1, \tag{10}$$

since A is necessarily positive definite. The matrix with perfect conditioning has PN = 1, that occurs in the case of orthogonal or diagonal matrices.

5.2. Weighted Graph and an Admissible Member

The relative stiffnesses (or flexibilities) of the members of a structure can be considered as positive integers associated with the members of the graph model of a structure, resulting in a *weighted graph* as

$$W(m_i) = \Sigma k_{ii} = 2(\alpha_1 + \alpha_4^z + \alpha_3^z), \tag{13}$$

where: $\alpha_1 = \dfrac{EA}{L}$, $\alpha_3^z = \dfrac{4EI}{L}$ and $\alpha_4^z = \dfrac{12EI}{L^3}$.

A different weight employing the square roots of the diagonal entries of $\mathbf{k}_{m_i}$ has also been used:

$$W(m_i) = \Sigma\sqrt{k_{ii}} = 2[(\alpha_1)^{1/2} + (\alpha_4^z)^{1/2} + (\alpha_3^z)^{1/2}]. \tag{14}$$

Other weight functions may be defined for representing the relative stiffness of the members of S, as appropriate.

Definition: Let the weight of members m_1, m_2, ..., $m_{M(S)}$ be defined by $W(m_1)$, $W(m_2)$,..., $W(m_{M(S)})$, respectively. A member m_i is called *F-admissible* if,

$$W(m_i) \geq \frac{1}{\alpha} \sum_{j=1}^{M(S)} \frac{W(m_j)}{M(S)}, \tag{15}$$

where α is an integer number which can be taken as 2,3,….Here α=2 is used; however, a complete study utilizing other values of α is required. If a member is not F-admissible, it is called F-*inadmissible* or *S-admissible*.

5.3. Optimally Conditioned Cycle Bases

In order to obtain optimally conditioned flexibility matrices, special static bases, correspondingly cycle bases possessing particular properties, should be selected.

A cycle basis is defined as an *optimally conditioned cycle basis* if:

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(a) It is an optimal cycle basis, i.e. the number of non-zero entries of the corresponding cycle adjacency matrix is minimum, leading to a maximal sparsity of the flexibility matrix.
(b)The members of greatest weight of S are included in the overlaps of the cycles; i.e. the off-diagonal terms of the corresponding flexibility matrix have the smallest possible magnitudes.

A weighted graph may have more than one optimal cycle basis. The one satisfying condition (b) is optimally conditioned. However, if no such a cycle basis exists, then a compromise should be found in satisfying conditions (a) and (b). In other words, a basis should be selected which partially satisfies both conditions. Since there is no algorithm for the formation of an optimal cycle basis, one should look only for a suboptimally conditioned cycle basis.

Example: Consider a 3×3 grid as shown in Fig. 16(a), with the relative weights of the members being encircled. An optimal cycle basis of S contains 9 regional cycles (mesh basis) and corresponds to:

$$L_T = \sum_{i=1}^{8} L(C^i \cap C_{i+1}) = 1+1+1+2+2+1+2+2 = 12 \cdot$$

The weight of the members contained in the overlaps is determined as,

$$W_T = \sum_{i=1}^{8} W(C^i \cap C_{i+1}) = 2+2+10+12+12+1+3+3 = 45,$$

where L_T and W_T are the *length* and *weight* of the overlaps of the selected cycles, respectively.

A suboptimal cycle basis of S is illustrated in Fig. 16(c) for which:

$$L'_{i=1} = \sum_{i=1}^{8} L(C^i \cap C_{i+1}) = 1+1+1+2+2+3+4+4 = 18.$$

The weight of the members contained in the overlaps is calculated as:

$$W'_{i=1} = \sum_{i=1}^{8} W(C^i \cap C_{i+1}) = 2+2+10+12+12+14+16+16 = 84.$$

The weight of the overlaps of the selected cycles is considerably increased at the expense of some increase of their lengths, hence some decrease in the sparsity of its cycle adjacency matrix. Obviously W_T can be further increased; however, the decrease of sparsity will significantly influence the optimality of the cycle basis.

In this structure, the members of weight 1 are inadmissible according to the definition of the previous section, since $1 < \frac{1}{2} \times \frac{69}{24} = 1.43$.

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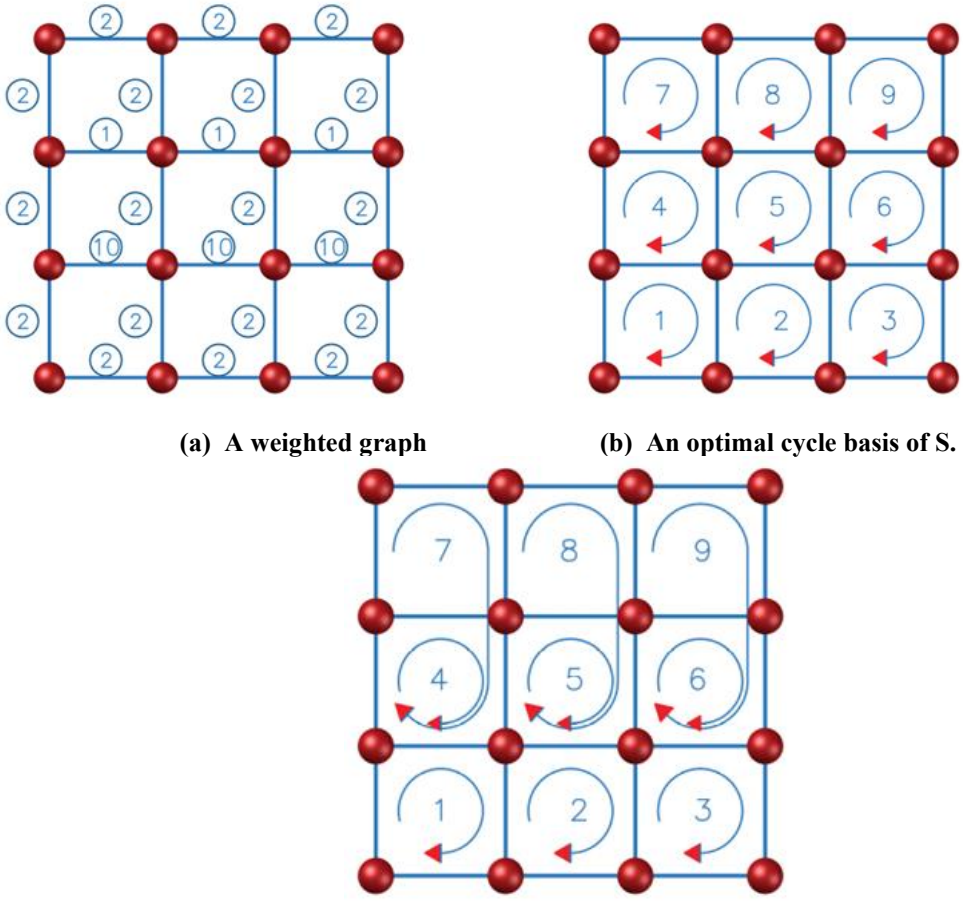

(a) A weighted graph **(b) An optimal cycle basis of S.**

(c) A suboptimally conditioned cycle basis

Figure16: A single-layer rigid-jointed grid S.

5.4. Formulation of the Conditioning Problem

The problem of selecting an optimally conditioned cycle basis can be formulated in the mathematical form as

$$\text{Min} \sum_{i=1}^{b_1(\bar{S})-1} L(C^i \cap C_{i+1}), \tag{16}$$

and

$$\text{Max} \sum_{i=1}^{b_1(\bar{S})-1} W(C^i \cap C_{i+1}), \tag{17}$$

Here $\bar{S}$ is obtained from S by a sequence of elementary contractions, where all the existing paths in S are replaced by single members [7] and $C^i = \bigcup_{j=1}^{i} C_j$.

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As can be observed, the problem is a multi-objective optimization problem, and the following algorithms are designed such that both objective functions are partially satisfied simultaneously, i.e. a compromise is obtained.

5.5. Suboptimally Conditioned Cycle Bases

In this section an algorithm is developed for the formation of suboptimally conditioned cycle bases of weighted graphs. On each selected cycle, for planar frames three, and for space frames six self-equilibrating stress systems are constructed. The condition number of the flexibility matrix corresponding to the selected static basis is obtained using the methods of Section 5.1.

Algorithm: This algorithm utilizes the chords of a special spanning tree to guaranty the independence of the selected cycles. To avoid the inclusion of inadmissible chords in the intersections of the cycles, such chords are not added to the set of members to be utilized in the formation of the cycles of S.

Step 1: Select the pseudo-centre "O" of S with a graph or algebraic graph theoretical method.

Step 2: Generate an SRT employing the members of highest weights, i.e.

2.1 Assign "1" to the other ends of all the members incident with O and

2.2 Find all members incident with nodes denoted by "1" and order them in ascending magnitude of their weights;

2.3 Select the tree members from the above ordered members, assigning "2" to the other ends.

Step 3: Repeat Step 2 as many times as needed until all the nodes of S are spanned and an SRT is formed.

Step 4: in this step the members incident with "1" should be ordered in ascending magnitude and the members of maximal weight are utilized as the chord of the first minimal length cycle. In case this chord is F-admissible, then it is added to the list of the tree members, and denoted by T^c.

Step 5: The second shortest length cycle should be generated on the second maximal weight member incident with "1" employing the members of T^c. Again add the chord to T^c if it is F-admissible. This process should be continued until all the chords which are incident with the nodes labelled as "1" are utilized.

Step 6: Repeat Steps 4 and 5 for all the nodes that are labelled by "2". Tthis process should be repeated sequentially for all the nodes labelled by 3, 4, ... , k, until a complete cycle basis is formed.

The above algorithm generates suboptimally conditioned cycle bases, and has the the following advantages compared to using fundamental cycle bases:

(a) The starting node is at the centre or pseudo-centre of S limits the length of the generated cycles.

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(b) Employing the utilized chords in the formation of cycles, further reduces the length of the selected cycles.
(c) Avoiding the addition of F-inadmissible chords results in the inclusion of weak members in the overlaps of the cycles.
(d) Using the members with the highest weight in each stage of forming an SRT results in the weaker members as candidates for chords, that can be excluded because of inadmissibility.

One can select a spanning tree of maximal weight employing the Greedy Algorithm in place of an SRT of maximal weight with respect to the centre node of S; however, in general, longer cycles will then be generated corresponding to a less sparse cycle adjacency matrix.

A further improvement can be achieved if comparison is made among the centre node (or nodes) and adjacent nodes to choose a node of higher average weight as a starting node. The average weight of a node is taken as the sum of the weights of the members incident with n_i / deg n_i. This improvement is because of including all the members of the root node in T^c.

For other efficient methods of selecting well-conditioned cycle basis one can refer to [19]. For the displacement method a special cut set basis will be needed as described in [20].

6. SWIFT ANALYSIS OF SYMMETRIC AND REGULAR STRUCTURES

For any given structure, if the corresponding structural matrices are well conditioned, or their study is ignored, then only sparsity and well-structuring become the criterions for optimality. In this case, the optimal analysis is termed as “a swift analysis”. Many efficient methods are presented for composition and decomposition of large and complex structures by Kaveh et al. [55]. In the later reference many efficient algorithms are provided for symmetric, near symmetric, and regular and near regular structures.

Symmetry is studied using canonical forms in matrices. Examples are studied in the work of Kaveh and Sayarinejad [56-58] for graph problems, dynamics and stability of frame structures in the work of Kaveh and Salimbahrami [59-61] and analysis of truss structures by Kaveh and Sharyari [62-64]. Ideas are generalized to general symmetries and regular graphs. In relation with this problem graph products has been very helpful, Imrich and Klavžar [65], Kaveh and Rahami [66], Kaveh and Dadfar [67] and Kaveh et al. [68]. Structures being out of symmetry by a small amount are extensively studied in Kaveh et al. [55].

The ideas are generalized to finite element models by Cassell [69]. Kaveh [70] Kaveh and Massoudi [71-73]. Many interesting stress-based finite element formulations are also developed in the work of Maunder and colleagues [74-76].

There are other scientific tools for swift analysis of structures. As an example group theory is used for symmetric structures. The interested reader may refer to Kaveh and Nikbakht [77-79] and Zingoni [80-82]. A combination of graph theory and group theory is also utilized to enhance the methods.

The early research by the present author has been affected by strong research which was performed in Sweden by Langefors [83-86], Samuelsson [87] and Wiberg [88]. The previous

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research of Russopoulos [89] at Greece followed by the extensive fundamental research of Argyris et al. [90] on finite element analysis have also been the other sources of motivations.

Graph theory is extensively used by Kron [91] in electrical engineering and some others in structural engineering [92-94]. Other important relevant references in structural analysis comprise of [95-98], and well-known books in finite elements methods by [99-101].

7. OPTIMAL ANALYSIS FOR OPTIMAL DESIGN OF STRUCTURES

Optimal analysis has many applications, however, the main application is in optimal design of structures where analysis must be performed hundred and sometimes thousands times. An example of this application can be found in the recent work of Kaveh and Zaerreza [102].

The displacement and force methods are the two well-known structural analyzing methods. The computational time required by these methods may enlarge by increasing the number of equations that must be solved to obtain the stress or displacement of the nodes. The number of equations depends on the degree of kinematical indeterminacy (DKI) and the degree of static indeterminacy (DSI). The DKI and DSI values represented the number of equations to be solved using the displacement and force methods, respectively. Although the time difference is not significant during a single analysis, the time gap grows over the optimization process owing to repeated structural analyses. Due to this computational overhead, researchers applied the force method instead of the displacement method when the optimization problem had less DSI than DKI. For example, Kaveh and Malakouti Rad [103] applied the force method for the optimum design of the structures using the hybrid genetic algorithm and particle swarm optimization. Kaveh and Rahami [104] applied the force method for the optimum design of truss structures.

Three improved algorithms named the E*nhanced Colliding Bodies Optimization (ECBO)*, the *Improved Shuffled Jaya Algorithm (IS-Jaya*), and the *Vibrating Particle Systems Statistical Regeneration Mechanism Algorithm (VPS-SRM)* are applied to the optimum design of the frame structures using the force method. The structures considered in this study have lower DSI than DKI, hence, the force method is faster than the displacement method. In addition, Kaveh and Zaerreza [102] demonstrate the effectiveness of the force method on the structures analyzed in this work.

Optimal analysis can also be used in the displacement method for optimal design of symmetric structures, e.g. the optimal design of cyclically symmetric domes with symmetric loading, where the multiplication of a slice of dome in cycle can be utilized for the analysis. This reduces the storage and computation time to a great extent, Kaveh [65] and Kaveh et al. [55].

8. CONCLUDING REMARKS

This paper is mainly devoted to the optimal analysis of skeletal structures. This review is intended to cover the force method, since reviews of the displacement method are extensively available.

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The sparsity of structural matrices is widely studied and the results are used for the formation of well-structured and well-conditioned matrices for frame structures. Here, the study is limited to a presentation of basic concepts and methods of skeletal structures. For efficient computation of frame analysis, different cycle bases are defined and efficient algorithms have been developed. However, for truss structures the formulation has been limited due to the absence of graph theoretical algorithms, and in such cases algebraic force methods has been used. A graph theoretical interpretation is only available for Gaussian elimination and it should be extended to other algebraic methods. This could result in an advantageous understanding of algebraic force methods.

Different ordering algorithms are available for the formation of the structural matrices with different patterns. However, efficient methods should be developed for the formation of well-conditioned structural matrices.

Optimal analysis is important for optimal design of structures, and is used both for the stiffness method analysis of cyclically symmetric structures like domes, and the force method analysis when the DSI of a structure is less than its DKI. Different applications can be extended to increase the efficiency of analysis and consequently the optimal design when metaheuristic algorithms are used.

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Appendix A

Basic Definitions from Theory of Graphs

Graph theory is a branch of mathematics started by Euler [108] as early as 1736. It took more than a hundred years before the second important contribution of Kirchhoff [26] had been made for the analysis of electric networks. It took another century before the first book was published by König [109]. After the Second world war, further books were published on graph theory by Ore [110], Tutte [111], Berge [112], Harary [113], and West [114], among many others.

For understanding this paper the reader can refer to concept and definitions from Refs. [113,114].

Appendix B

Expansion process is very important and one can find the property of a model in different stages of the process of expansion by the following theorem. One can also select subgraphs of specified properties such as cycles, γ-cycles and substructures. Similar theorem exists in the algebraic topology (Maunder [115]), however the considered function in the following theorem is more general than the first Betti number.

Union-Intersection Theorem (Kaveh [4,7]): Consider S as the union of q subgraphs S_1, S_2, S_3, ... , S_q, where the following functions are defined:

$$\nu(S) = aM(S) + bN(S) + c\nu_0(S),$$

$$\nu(S_i) = aM(S_i) + bN(S_i) + c\nu_0(S_i) \qquad i = 1,2,...,q$$

$$\nu(A_i) = aM(A_i) + bN(A_i) + c\nu_0(A_i) \qquad i = 2,3,...,q,$$

Here, $A_i = S^{i-1} \cap S_i$ and $S^{i-1} = S_1 \cup S_2 \cup ... \cup S_{i-1}$ and $\nu(S)$ is a function defining the degree of static indeterminacy $\gamma(S)$ or kinematical indeterminacy $\eta(S)$ for a structure S. It can easily be proved that

$$[\nu(S) - c\nu_0(S)] = \sum_{i=1}^{q}[\nu(S_i) - c\nu_0(S_i)] - \sum_{i=2}^{q}[\nu(A_i) - c\nu_0(A_i)]. \qquad \text{(B-1)}$$

Special Case: In the expansion process if S and each considered subgraph (S_i for i = 1,...,q) is connected (non-disjoint), then Eq. (B-1) will be simplified as,

$$\nu(S) = \sum_{i=1}^{q}\nu(S_i) - \sum_{i=2}^{q}\bar{\nu}(A_i), \qquad \text{(B-2)}$$

In this relation

$$\bar{\nu}(A_i) = aM(A_i) + bN(A_i) + c.$$

In order to calculate the DSI and DKI of a complex structure or a structure having a large number of members, one prefers to select a repeated unit of the structure and joins these units sequentially in a connected form. Therefore, Eq. (B-2) will be applicable in place of Eq. (B-1) for obtaining the overall property of the structure.

Compliance with ethical standards

Conflict of interest: No potential conflict of interest was reported by the author.

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