ISOGEOMETRIC TOPOLOGY OPTIMIZATION OF STRUCTURES BY USING MMA

S. M. Tavakkoli[*,†,a], B. Hassani[b] and H. Ghasemnejad[a]
[a]Department of Civil Engineering, Shahrood University of Technology, Shahrood, Iran
[b]Department of Mechanical Engineering., Ferdowsi University of Mashhad, Mashhad, Iran

ABSTRACT

The Isogeometric Analysis (IA) method is applied for structural topology optimization instead of the finite element method. For this purpose, the material density is considered as a continuous function throughout the design domain and approximated by the Non-Uniform Rational B-Spline (NURBS) basis functions. The coordinates of control points which are also used for constructing the density function, are considered as design variables of the optimization problem. In order to change the design variables towards optimum, the Method of Moving Asymptotes (MMA) is used. To alleviate the formation of layouts with porous media, the density function is penalized during the optimization process. A few examples are presented to demonstrate the performance of the method.

1. INTRODUCTION

Structural topology optimization methods have made remarkable progress in recent years. This kind of optimization is employed mainly to specify the optimum number and location of holes in the configuration of the structure which is usually followed by shape optimization in order to find optimum boundaries. Topology optimization has received enormous attention since the introduction of the homogenization approach to topology optimization by Bendsøe and Kikuchi [1] but its origin goes back to the minimum weight

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[*] Corresponding author: S. M. Tavakkoli, Department of Civil Engineering, Shahrood University of Technology, Shahrood, Iran
[†]E-mail address: mtavakkoli@shahroodut.ac.ir (S. M. Tavakkoli)

structures of Michell [2].

To solve the topology optimization problem several methods such as optimality criteria (OC) methods [3,4], the approximation methods [5-7], CONLIN [8] and the Method of Moving Asymptotes (MMA) [9], even more heuristic methods such as genetic algorithm [10-12] and Ant colony [13] are employed. Also, less mathematically rigorous methods such as the evolutionary structural optimization method (ESO) can be named [14]. Moreover, several different approaches are devised that use the level set methods [15-18]. As an optimization engine the MMA method is used here which has been proven to be amongst the most effective to solve the topology optimization problems.

Form another view, the topology optimization methods can be divided into two types, element based and nodal based. In structural topology design the optimum distribution of a given material, mostly isotropic, in a defined domain is searched [19,20]. The material assigned to each point of the design domain can be specified by a material distribution function. In the element based approaches the material density function is constant within each finite element. However, in the nodal based methods which have more recently been introduced by Belytschko *et al* [18], the material densities are determined at the discretization nodes or points. The nodal based view of topology optimization is more suitable when meshless methods are employed for structural analysis [21]. Recently, by using this approach the IA method has also been utilized for topology optimization [22].

The IA method has been proposed and developed by Hughes and his co-workers in recent years [23-27]. This method is a logical extension and generalization of the classical finite element method and has many features in common with it. However, it is more geometrically based and takes inspiration from Computer Aided Geometry Design (CAGD). A primary goal of IA is to be geometrically precise no matter how coarse the discretization beside simplification of mesh refinement by eliminating the need for communication with the CAD geometry once the initial model is constructed. The main idea of the method is to use the same basis functions which are employed for geometry description for approximation and interpolation of the unknown field variables as well. Due to some interesting properties of B-splines and NURBS, they are perfect candidates for this purpose.

The IA method has been utilized for topology optimization of structures instead of the FE approach; where the NURBS basis functions have also been used for approximating the material density function by Hassani *et al* [22]. The OC method was used and a new updating scheme was presented in order to solve the optimization problem.

The method presented in this article falls within the category of nodal based methods which uses control points instead of nodal points and employs the recently developed IA approach [22]. The material distribution function is approximated over the whole domain and is restricted to be within zero (for empty areas or voids) and one (for solid areas) interval. Also, similar to the SIMP method a penalty exponent is implemented to suppress formation of undesirable porous media inside the optimal layout. In this research, the gradient based MMA method is employed to solve the optimization problem and therefore the sensitivity analysis is also carried out. Although the MMA is a mathematical based method, it is able to handle large scale optimization problems even with different type of shape and topology design variables, simultaneously [28].

The outline of this paper is as follows. In Section 2, the IA method for plane stress

problems is briefly explained. Section 3 is devoted to the concise definition of topology optimization problem. Also a material distribution function is considered to define the topology of the structure. In Section 4 the sensitivity analysis is done. Three examples are presented in section 5 to demonstrate the performance of the method. Finally, the results are discussed in the last section.

2. ISOGEOMETRICAL ANALYSIS

By recent developments in the CAGD technology, the geometrical definition and generation of complex surfaces and objects have become achievable [29]. For this purpose, Splines and some modified versions of them, i.e. NURBS and T-Splines, are commonly employed. The main idea in the IA method is that any component of a field variable which satisfies a governing partial differential equation, i.e. the solution, is imagined as a surface that can be constructed by the proper versions of splines [30, 31]. For example, in displacement method for elasticity problems, each of the components of the displacement vector is considered as a surface which can be constructed by NURBS and the defining parameters of these surfaces are sought. The criteria for finding these parameters can be obtained by minimizing a total potential energy functional or equivalently by implementation of the virtual work principle.

The IA method has some features in common with other numerical methods such as finite elements and meshfree methods. Discretization of the domain of interest is performed by using the control points of splines instead of, for instance, using the finite element meshes, finite difference grids, or collection of points in the meshfree methods. Also, the basis functions of these splines are used both for approximation of the unknown variables as well as for interpolation.

The procedure of the IA for elasticity problems is comprised of the following steps. First, the geometry of the domain of interest is constructed by using the NURBS technology. Depending on the complexity of the geometry and topology of the problem, multiple NURBS patches can be used in this stage. These patches may be thought of as kind of macro elements in the finite element method and can be assembled in the same fashion [23]. In the next step, borrowing the ideas from isoparametric finite elements, the geometry as well as the displacement components are approximated by making use of the NURBS basis functions. Then, following a standard procedure like the weighted residuals or the variational methods, or similarly using the principle of virtual work, the approximated quantities are substituted into the obtained relations. This will result in a system of linear equations to be solved. One should note that following this procedure the control variables are evaluated and to obtain the displacements at certain points a kind of post processing is required. A brief introduction to the construction of NURBS surfaces followed by derivation of IA formulation for plane elasticity problems are the subjects of the next two subsections.

2.1. Surface and volume definition by NURBS

A NURBS surface is parametrically constructed as [29]

$$S(r,s) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \frac{N_{i,p_1}(r) N_{j,p_2}(s) \, \omega_{i,j}}{\sum_{k=0}^{n_1} \sum_{l=0}^{n_2} N_{k,p_1}(r) N_{l,p_2}(s) \, \omega_{k,l}} P_{i,j} \tag{1}$$

where $P_{i,j}$ are $(n_1+1) \times (n_2+1)$ control points, $\omega_{i,j}$ are the associated weights, and $N_{i,p_1}(r)$ and $N_{j,p_2}(s)$ are the normalized B-splines basis functions of degree p_1 and p_2, respectively. The i-th B-spline basis function of degree p_1, denoted by $N_{i,p_1}(r)$, is defined recursively as:

$$N_{i,0}(r) = \begin{cases} 1 & if \quad r_i \le r \le r_{i+1} \\ 0 & otherwise \end{cases}$$

$$N_{i,p_1} = \frac{r - r_i}{r_{i+p_1} - r_i} N_{i,p_1-1}(r) + \frac{r_{i+p_1+1} - r}{r_{i+p_1+1} - r_{i+1}} N_{i+1,p_1-1}(r) \tag{2}$$

where $\mathbf{r} = \{r_0, r_1, ..., r_{m_1}\}$ is the knot vector and r_i are a non-decreasing sequence of real numbers, which are called knots. The knot vector $\mathbf{s} = \{s_0, s_1, ..., s_{m_2}\}$ is employed to define the $N_{j,p_2}(s)$ basis functions for other direction. The interval $[r_0, r_{m_1}] \times [s_0, s_{m_2}]$ forms a patch [23]. A knot vector, for instance in r direction, is called open if the first and last knots have a multiplicity of p_1+1. In this case, the number of knots is equal to $m_1 = n_1 + p_1 + 1$. Also, the interval $[r_i, r_{i+1})$ is called a knot span where at most p_1+1 of the basis functions $N_{i,p_1}(r)$ are non-zero which are $N_{i-p_1,p_1}(r)$, ..., $N_{i,p_1}(r)$. For more details Reference [29] can be consulted.

2.2. Numerical formulation for plane elasticity problems

In the IA method, the domain of problem might be divided into subdomains or patches so that B-spline or NURBS parametric space is local to these patches. A patch is like an element in the finite element method and the approximation of unknown function can be written over a patch. Therefore, the global coefficient matrix, which is similar to the stiffness matrix in elasticity problems, can be constructed by employing the conventional assembling which is used in the finite element method.

By using the NURBS basis functions for a patch p, the approximated displacement functions $\mathbf{u}^p = [u, v]$ can be written as

$$\mathbf{u}^p(r,s) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} R_{i,j}(r,s) \mathbf{u}_{i,j}^p \tag{3}$$

where $R_{i,j}(r,s)$ is the rational term in Equation (1). It should be noted that the geometry is

also approximated by B-spline basis functions as

$$\mathbf{x}^p(r,s) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} R_{i,j}(r,s)\mathbf{x}_{i,j}^p \tag{4}$$

By using the local support property of NURBS basis functions, the above relation can be summarized as it follows in any given $(r,s) \in [r_i, r_{i+1}) \times [s_j, s_{j+1})$.

$$\mathbf{u}^p(r,s) = \sum_{e=i-p_1}^{i} \sum_{f=j-p_2}^{j} R_{e,f}(r,s)\mathbf{u}_{e,f}^p = \mathbf{R}\mathbf{U} \tag{5}$$

$$\mathbf{x}^p(r,s) = \sum_{e=i-p_1}^{i} \sum_{f=j-p_2}^{j} R_{e,f}(r,s)\mathbf{x}_{e,f}^p = \mathbf{R}\mathbf{X} \tag{6}$$

The strain-displacement matrix $\mathbf{B}$ can be constructed from the following fundamental equations

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u} \;\rightarrow\; \boldsymbol{\varepsilon} = \mathbf{B}\mathbf{U} \tag{7}$$

where $\mathbf{D}$ is the differential operation matrix. Following a standard approach for the derivation of the finite elements formulation, the matrix of coefficients can easily be obtained. For example, by implementing the virtual displacement method with existence of body forces $\mathbf{b}$ and traction forces $\mathbf{t}$ we can write

$$\int_{V^p} \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV - \int_{V^p} \delta\mathbf{u}^T \mathbf{b}\, dV - \int_{\Gamma^p} \delta\mathbf{u}^T \mathbf{t}\, d\Gamma = 0, \tag{8}$$

where V^p and Γ^p are the volume and the boundary of patch p.

Now, by substituting $\delta\boldsymbol{\varepsilon} = \mathbf{B}\delta\mathbf{U}$ from Equation (7) and the constitutive equation $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$, in Equation (8) and by dropping the non-zero coefficient of $\delta\mathbf{U}^T$, the matrix of coefficients can be obtained as

$$\mathbf{K}^p = \int_{V^p} \mathbf{B}^T \mathbf{C}\mathbf{B}\, dV \tag{9}$$

3. TOPOLOGY OPTIMIZATION PROBLEM

The problem at hand is defined as finding the stiffest possible structure when a certain amount of material is given. A structure with maximum global stiffness provides a minimum for the external work with the real displacement field or minimum mean

compliance. Therefore, the topology optimization problem can be constructed as below [19,20]

$$
\begin{aligned}
\min \quad & \mathbf{f}^T \mathbf{u} \\
subject\ to: \quad & equilibrium \\
& \Omega_s \leq \bar{\Omega}_s \\
& design\ restrictions
\end{aligned}
\tag{10}
$$

where $\mathbf{u}$ and $\mathbf{f}$ are the displacement and load vectors and $\bar{\Omega}_s$ is the amount of material available. Ω_s is the volume of solid material in each design.

In structural topology optimization, the problem is how to distribute the material in order to minimize the objective function. In other words, the goal can be thought of as determination of the optimal spatial material distribution and can be described as a density function $\phi(\mathbf{x})$ for every point of the design domain.

$$
\phi(\mathbf{x}) = \begin{cases} 1 & if \quad \mathbf{x} \in \Omega_s \\ 0 & if \quad \mathbf{x} \in \Omega \setminus \Omega_s \end{cases}
\tag{11}
$$

The above material distribution function can be approximated by the NURBS basis functions over a patch.

$$
\phi^p(r,s) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} R_{i,j}(r,s) \Phi^p_{i,j}
\tag{12}
$$

where $\Phi^p_{i,j}$ are control points of the NURBS surface in patch p and can be assumed as design variables of the optimization problem. It is noted that the same basis functions are used for approximation of geometry, displacements and the density function.

Inspired by the SIMP method, in order to prevent the porous area, the density function is penalized for evaluating the artificial elasticity matrix. Therefore,

$$
\begin{aligned}
\rho(\mathbf{x}) &= \varphi(\mathbf{x}) \rho^0 \\
\mathbf{C}(\mathbf{x}) &= \varphi(\mathbf{x})^\mu \mathbf{C}^0
\end{aligned}
\tag{13}
$$

where ρ^0 and $\mathbf{C}^0$ are the density and elasticity matrix of solid material, respectively. It is noted that the density of used material is not penalized to make the material volume constraint real. The discretized optimization problem can be written as below:

$$
\begin{aligned}
&\min_{\Phi_{i,j}} && \mathbf{f}^T \mathbf{u} \\
&subject\ to: && \mathbf{K u} = \mathbf{f} \\
& && \Phi_{i,j} - 1 \le 0 \quad i = 1,...,n_1, \quad j = 1,...,n_2 \\
& && -\Phi_{i,j} \le 0 \quad i = 1,...,n_1, \quad j = 1,...,n_2 \\
&and && \Omega_s \le \bar{\Omega}_s
\end{aligned}
\tag{14}
$$

where according to equations (9) and (13) the stiffness matrix $\mathbf{K}$ can be written as follows:

$$
\mathbf{K} = \sum_{p=1}^{np} \mathbf{K}^p = \sum_{p=1}^{np} \int_{\Omega^p} \mathbf{B}^T \phi^\mu \mathbf{C}^0 \mathbf{B}\, d\Omega
\tag{15}
$$

where np is the number of patches in the design domain which are assembled similar to the FE method.

4. SENSITIVITY ANALYSIS

In this research MMA is utilized to solve the topology optimization problem. This algorithm has shown promising results in large scale topology optimization problems [20]. In order to use gradient based MMA algorithm the derivatives of the objective function and the constraints with respect to the design variables are required as the sensitivity information. Assumed that $\ell(\phi)$ denotes the objective function we have:

$$
\ell(\phi) = \mathbf{f}^T \mathbf{u}
\tag{16}
$$

where displacement field $\mathbf{u}$ satisfies the equilibrium $\mathbf{K}(\varphi)\mathbf{u} = \mathbf{f}$. Here, the adjoint method is used where the derivatives of the displacement are not calculated explicitly. To achieve this, function $\ell(\phi)$ is rewritten by adding a zero term due to equilibrium [20]:

$$
\ell(\phi) = \mathbf{f}^T \mathbf{u} - \delta \mathbf{u}^T (\mathbf{K u} - \mathbf{f})
\tag{17}
$$

where $\delta \mathbf{u}$ is any arbitrary but fixed real vector. Derivatives of the function with respect to the design variables after some manipulations are obtained as follows:

$$
\frac{\partial \ell(\phi)}{\partial \phi} = (\mathbf{f}^T - \delta \mathbf{u}^T \mathbf{K}) \frac{\partial \mathbf{u}}{\partial \phi} - \delta \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial \phi} \mathbf{u}
\tag{18}
$$

From this, when $\delta \mathbf{u}$ satisfies the adjoint equation ($\mathbf{f}^T - \delta \mathbf{u}^T \mathbf{K} = 0$), the above equation

can be written as:

$$\frac{\partial \ell(\phi)}{\partial \phi} = -\delta \mathbf{u}^T \frac{\partial \mathbf{K}}{\partial \phi} \mathbf{u} \tag{19}$$

Therefore, for the real state of displacement $\delta \mathbf{u} = \mathbf{u}$ and from equations (15) and (19) the derivative of the objective function with respect to the design variables can be written as follows:

$$\frac{\partial \ell(\phi)}{\partial \phi} = -\mathbf{u}^T \frac{\partial \mathbf{K}}{\partial \phi} \mathbf{u} = -\mathbf{u}^T \left(\sum_{p=1}^{np} \int_{\Omega^p} (\mu \phi^{\mu-1}) \mathbf{B}^T \mathbf{C}^0 \mathbf{B} \, d\Omega \right) \mathbf{u} \tag{20}$$

5. NUMERICAL EXAMPLES

In this section we present three examples to illustrate the performance of the method. For the sake of comparison the examples are taken from our previous work [22] where IA and OC methods were used for analysis and topology optimization of the structures, respectively. In all examples the modulus of elasticity and the Poisson's ratio are considered as 1500 kgf/cm^2 and 0.3, respectively. Also, the exponent $\mu = 3$ is considered for penalizing the density function. In all of the following examples dimensions are assumed as $L = 8\,cm$ and $H = 5\,cm$. Also, the point load magnitude is taken as $P = 100\,kgf$.

5.1. Example 1

In this example, the effect of different control points by using OC and MMA methods is studied. Here a short cantilever beam subjected to a point load at the bottom corner is considered (Figure 1(a)). In all of the discretizations, equally spaced open knot vectors are used for each direction. The considered knot vectors are given in Table 1. For instance the control points arrangement is also shown in Figure 1(b). The volume fraction in all cases is taken as $\Omega_s / \Omega_{total} = 40\%$.

Table 1. Considered knot vectors for each patch

No. of Control Points	No of Patches	The employed equally spaced knot vectors
187	1	$\mathbf{r} = \{0, 0, 0, 0.066, ..., 0.933, 1, 1, 1\}$, $\mathbf{s} = \{0, 0, 0, 0.111, ..., 0.888, 1, 1, 1\}$ *for* $p_1 = p_2 = 2$ $\mathbf{r} = \{0, 0, 0, 0.071, ..., 0.929, 1, 1, 1\}$, $\mathbf{s} = \{0, 0, 0, 0.125, ..., 0.875, 1, 1, 1\}$ *for* $p_1 = p_2 = 3$
1617	96	$\mathbf{r} = \{0, 0, 0, 0.333, 0.666, 1, 1, 1\}$, $\mathbf{s} = \{0, 0, 0, 0.333, 0.666, 1, 1, 1\}$ *for* $p_1 = p_2 = 2$ $\mathbf{r} = \{0, 0, 0, 0, 0.5, 1, 1, 1, 1\}$, $\mathbf{s} = \{0, 0, 0, 0, 0.5, 1, 1, 1, 1\}$ *for* $p_1 = p_2 = 3$

The obtained results by using 187 control points with the MMA and OC algorithms are depicted in Figures 1(c) and (d), respectively. In this case the results are similar to each

other and as it is expected the porous media has emerged due to relatively small number of control points. Figures 1(e) and 1(f) are obtained by using 1617 control points with MMA and OC methods respectively. By increasing the number of control points layouts with less intermediate material densities, i.e. gray areas, are obtained. On the other hand, the results from MMA are more sensitive to the number of the discretizing points and different topology comes out which is a well known difficulty in element based method especially when lower order FE are used [32]. However it is noticed that the values of objective functions in the last iteration are approximately the same. The density function of Figure 1(e) is shown in Figure 2 in 3D.
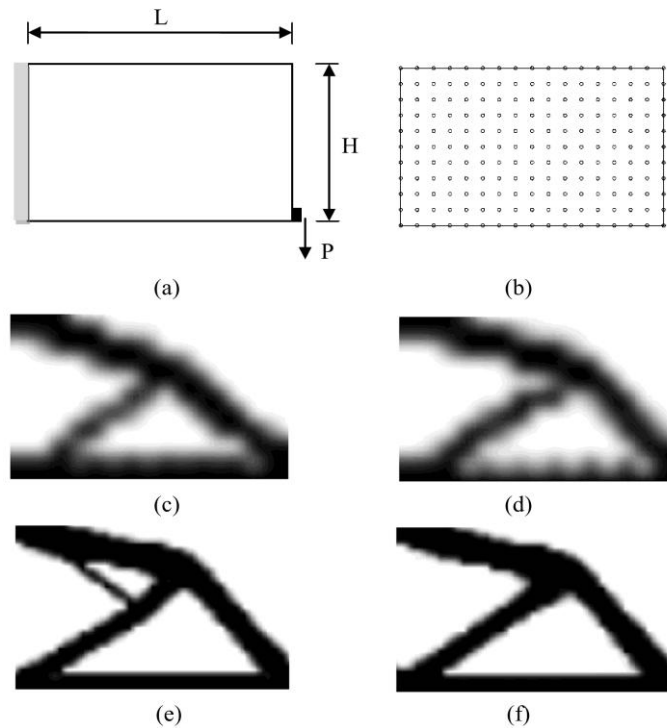

Figure 1. Short cantilever beam: (a) problem definition; (b) control net with 17 * 12 =187 points; (c), (d) optimum topologies by using 187; (e), (f) 1617 control points for MMA and OC methods, respectively.
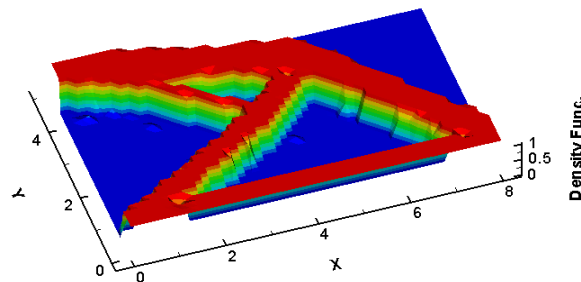

Figure 2. Optimum density function in 3D by using 1617 control points

Figure 3 illustrates the iteration history of the strain energies in example 1. It is experienced that the objective function is converged more rapidly by using MMA than OC method. However, the MMA is slower than OC method in each iteration because of being a mathematical gradient based approach
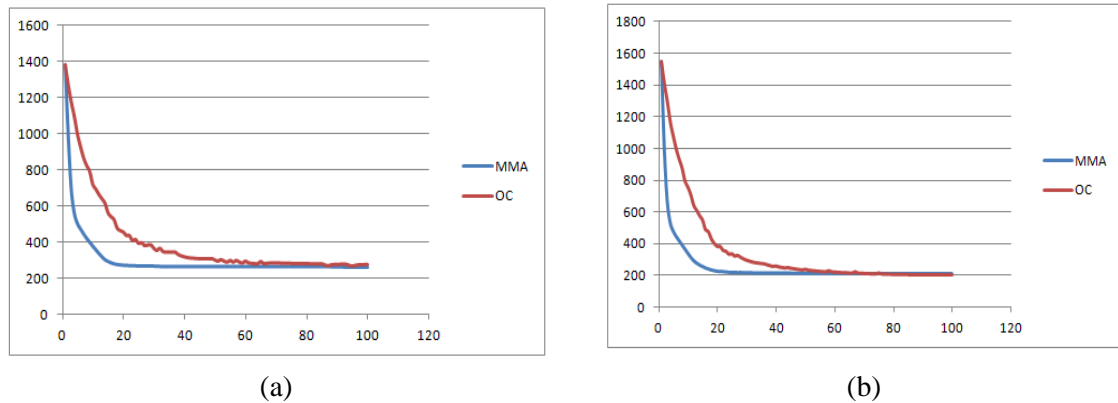

(a)   (b)

Figure 3. Iteration history for example 1 (a) 187 (b) 1617 control points

5.2. Example 2

In this example the ability of the proposed method in capturing the optimum topology and the effect of the number of control points is studied. For this purpose, a couple of control nets with 400 and 1617 points are used for discretizing the design domain as well as the material density function. The geometry, loading and boundary conditions are illustrated in Figure 4. The degree of the NURBS' basis functions is assumed $p = 2$. The resulted layouts from MMA are depicted in Figures 5(a) and 5(c). Figure 5(b) and 5(d) illustrates the results of OC method. As it is observed, the topologies are the same. Also the number of control points has not changed the topology. The optimal design form Sigmund's 99 line MATLAB code (Sigmund 2001) is illustrated in Figure 6. It is shown that the obtained topologies in both methods are identical.
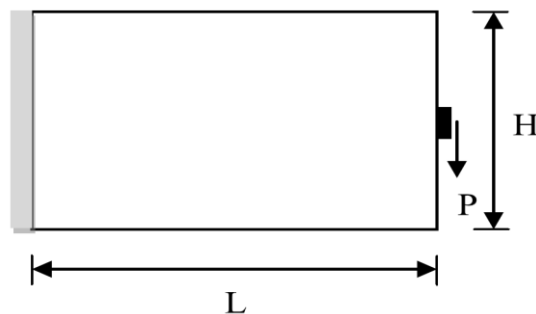

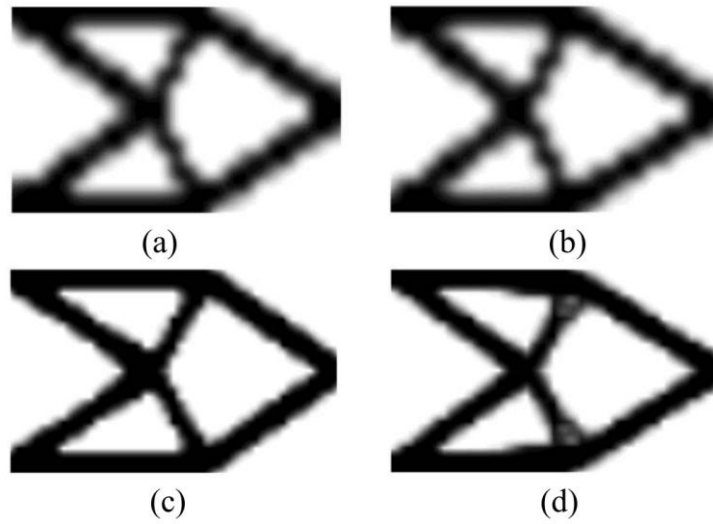Figure 4. Problem definition of example 2

Figure 5. Optimum layout by using (a),(c) MMA with 400 and 1617 control points (b),(d) OC with 400 and 1617 control points

Figure 6. Optimum layout by SIMP and FEM [33]

5.3. Example 3

The value of the specified material volume in the design domain can affect the optimal topology. To demonstrate this, a short beam problem is solved with three different volume fractions 20%, 30% and 50%. The design domain, supports of beam and loading are shown in Figure 7. In this example, 96 patches with 1617 control points are used for discretizing the design domain and the degree of NURBS' basis functions is 2. The employed knot vectors are the same as Example 1. The results are illustrated in Figure 8.
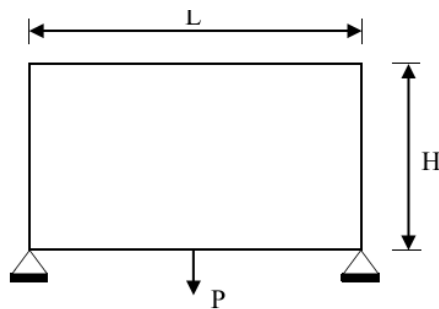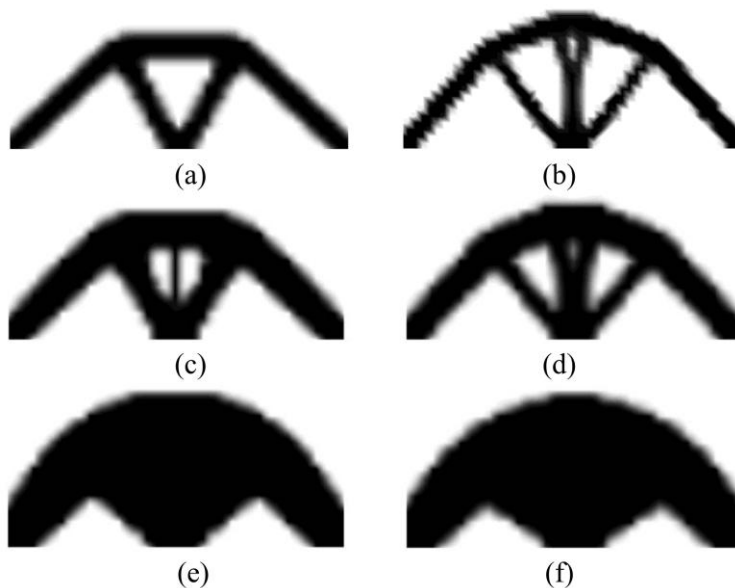
Figure 7. Problem definition of example 3

(a)   (b)

(c)   (d)

(e)   (f)

Figure 8. Optimum layouts with (a),(b) $\Omega_S/\overline{\Omega}s = 20\%$ (c),(d) $\Omega_S/\overline{\Omega}s = 30\%$ (e),(f) $\Omega_S/\overline{\Omega}s = 50\%$ by using MMA and OC method, respectively

6. CONCLUSION

In this research, a control point-based approach is presented for structural topology optimization when the IA method is used for analysis of the plane elasticity problem. A material distribution function is introduced and approximated by NURBS' basis functions. This enables us to change the topology by relocating the discretizing control points of the material distribution function. The MMA method is used to solve the optimization problem and for this purpose the sensitivity analysis is carried out. It is experienced that MMA is converged in fewer iterations than OC method and well suited for dealing with isogeometric topology optimization problem. The results are checkerboard free but, not totally mesh independent and can change by using more discretization control points. Therefore, further research is needed to remove the instability of dependency to the number of discretizing points.

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